THE

ELEMENTS OF EUCLID,

BOOKS I.—VI.; XI. 1—21; XII. 1, 2;

A New Text,

BASED ON THAT OF SIMSON.

EDITED BY

HENRY J. HOSE, B.A.

MATHEMATICAL MASTER OF WESTMINSTER SCHOOL,

AND

FORMERLY SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE.

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TO

ROBERT PITT EDKINS, ESQ., M.A.

PROFESSOR OF GEOMETRY IN GRESHAM COLLEGE,

AND SECOND MASTER OF THE CITY OF LONDON SCHOOL,

THIS EDITION OF

THE ELEMENTS OF EUCLID,

IS RESPECTFULLY DEDICATED,

BY

HIS OLD PUPIL.

PREFACE.

Professor De Morgan, in speaking of Simson's Edition of the Elements (Dictionary of Greek and Roman Biography and Mythology, s. v. Eucleides), says, "with the exception of the editorial fancy about the perfect restoration of Euclid, there is little to object to in this celebrated edition. It might indeed have been expected that some notice would have been taken of various points on which Euclid has evidently fallen short of that formality of rigor which is tacitly claimed for him." In preparing an edition for the use of schools and those commencing the study of geometry, it has been the Editor's aim to restore such "formality of rigor" in all places where it seemed wanting, and to render both text and figures as accurate as he could. No step has accordingly been omitted in the propositions, or left implied; the text has been made clearer and more

symmetrical by marking the divisions into cases, and stating similar pieces of reasoning as far as may be in the same words; all looseness of expression (in the enunciations, for example, or the determining of points, etc. in the figures) has been carefully corrected; and a new set of figures drawn, the thick lines of which are those that are given in the enunciation of a proposition, and the thin such as are afterwards made use of in the construction or proof.

GREAT DEAN'S YARD,
WESTMINSTER,
23rd Feb. 1853.

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THE

ELEMENTS OF EUCLID.

BOOK I.

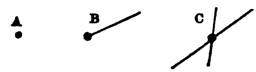
DEFINITIONS.

I.

A POINT is that which has neither length, breadth, nor thickness, but position only.

OBS. A point is usually denoted by a single capital letter of the alphabet.

Ex. We speak of the point A, the point B, the point C.



II.

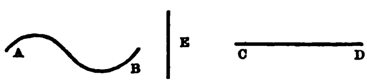
A line is that which has neither breadth nor thickness, but length only.

III.

The extremities of a line are points.

OBS. A line is usually denoted by two letters, these two letters denoting the points which are the extremities of the line. But lines are sometimes denoted by a single letter.

Ex. We speak of the lines AB, CD; A, B denoting the points which are the extremities of one line,



and C, D those which are the extremities of the other. We may also speak of the line E.

IV.

A straight line is a line which lies evenly between its extreme points.

V.

A superficies or surface is that which has length and breadth, but not thickness.

VI.

The extremities of a surface are lines.

VII.

A plane surface or plane is a surface, in which, if any two points be taken, the straight line of which they are the extremities, lies wholly in that surface, i. e. every point in the straight line is also a point in the surface.

VIII.

A plane angle is the inclination of two lines to one another in a plane, which meet together in a point, but are not in the same direction.

IX.

A plane rectilinear angle is the inclination of two straight lines to one another, which meet together in a point, but are not in the same straight line.

Obs. 1. Unless the contrary be expressly stated, whenever an angle is spoken of, a plane rectilinear angle is to be understood.

Obs. 2. When there are several angles at one point, any one of them is denoted by three letters, of which the letter put between the other two denotes the point where the straight lines meet together; and one of these two denotes some point in one of the two straight lines, that include the angle, and the other denotes some point in the other, the order of the first and third being indifferent.

Ex. Of the three angles at the point A (Fig. 1), the angle included by the straight lines AC, AB is denoted by CAB or BAC; the

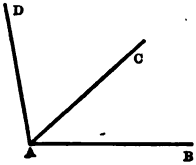


Fig. 1.

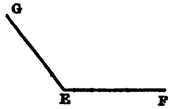


Fig. 2.

angle included by the straight lines AC, AD is denoted by CAD or DAC; and the angle included by the straight lines AB, AD is denoted by BAD or DAB.

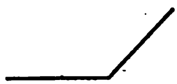
Obs. 3. But when there is only one angle at a point, it may be denoted either by the single letter that denotes the point, or by three letters as above.

Ex. The angle at the point E (Fig. 2) may either be denoted by E, or by FEG, or by GEF.

X.

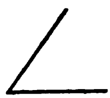
When one straight line standing on another straight line makes with that straight line, or with that straight line produced, if necessary, the adjacent angles equal to one another, each of these angles is defined to be a right angle; and each of the straight lines is defined to be perpendicular to the other.

XI.



An obtuse angle is an angle which is greater than a right angle.

XII.



An acute angle is an angle which is less than a right angle.

XIII.

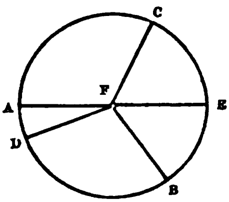
The boundary is the extremity of any thing.

XIV.

A figure is a portion of space which is enclosed by one or more boundaries; and when all the points in a figure are also points in the same plane, the figure is called a plane figure.

Obs. Unless the contrary be expressly stated, whenever a figure is spoken of a plane figure is to be understood.

XV.



A circle is defined to be a plane figure, contained by one line, such that all straight lines drawn from a certain point within the figure, to meet this one line, are equal to one another.

XVI.

This one line is called the circumference of the circle; the point within the figure is called the centre of the circle; and any one of the straight lines drawn from the centre to meet the circumference is called a radius of the circle.

OBS. It is usual to denote a circle by three letters, these three letters denoting any three points in the circumference of the circle; and the same three letters may be taken to denote the circumference.

Ex. In the above figure the circle may be denoted by ABC, or by DBC, or by BEA, &c.; and we may speak of the circumference ABC, or DBC, &c. F is the centre; and FA, FB, FC, &c., are radii.

XVII.

A diameter of a circle is any straight line drawn through the centre and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by any diameter of a circle, and by either of the two parts of the circumference, into which it is divided by the diameter.

XIX.

This definition is the same as Bk. iii. Def. 6.

XX.

When the lines which contain a plane figure are all straight lines, it is called a rectilinear figure, and the straight lines are called its sides.

XXI.

A rectilinear figure which has three sides is called a triangle.

XXII.

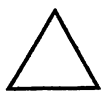
A rectilinear figure which has four sides is called a quadrilateral figure; one which has five, a pentagon; and one which has six, a hexagon.

XXIII.

Polygon is the general name for a rectilinear figure of any number of sides, including the triangle, quadrilateral figure, &c., as particular cases.

XXIV.

An equilateral triangle is a triangle which has its three sides all equal; an equilateral polygon is a polygon with all its sides equal; and an equiangular polygon is a polygon with all its angles equal.



XXV.

An isosceles triangle is a triangle which has two of its three sides equal.

XXVI.

A scalene triangle is a triangle which has its three sides all unequal.

XXVII.

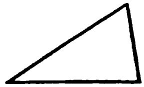
A right-angled triangle is a triangle one of the three angles of which is a right angle.

XXVIII.

An obtuse-angled triangle is a triangle one of the three angles of which is an obtuse angle.

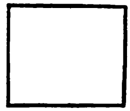
XXIX.

An acute-angled triangle is a triangle each of the three angles of which is an acute angle.



XXX.

A square is defined to be a four-sided figure which has all its sides equal and all its angles right angles.

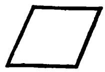


XXXI.

An oblong is a four-sided figure which has all its angles right angles, but has not all its sides equal.

XXXII.

A rhombus is a four-sided figure which has all its sides equal, but its angles not right angles.



XXXIII.

A rhomboid is a four-sided figure which has its opposite sides equal, but all its sides are not equal, and its angles are not right angles.

XXXIV.

Trapeziums are such four-sided figures as are not included in the four preceding definitions.

XXXV.

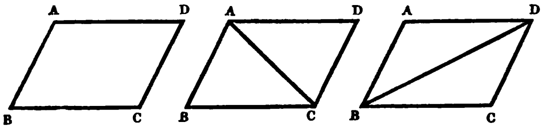
Parallel straight lines are such straight lines as are in the same plane, and as, being produced ever so far both ways, do not meet.

XXXVI.

A parallelogram is a four-sided figure which has its opposite sides parallel; and each of the two

straight lines drawn joining the two pairs of opposite angular points, is called a diagonal of the parallelogram.

OBS. In the figure the parallelogram is denoted either by ABCD, or by AC, or by BD; AC, which joins the opposite angular points A, C, is one diagonal; and BD, which joins the opposite angular points B, D, is the other diagonal.



This def^{n}., and Bk. ii. Def. 1, have superseded Def^{s}. 31, 32, 33.

POSTULATES.

I.

Let it be granted that a straight line may be drawn from any one given point to any other given point.

II.

Let it be granted that a given terminated straight line may be produced to any length required either way in a straight line.

III.

Let it be granted that a circle may be described from any given point as centre, at any given distance from that centre; or, what is the same thing, with any given point as centre, and with any given finite straight line drawn from that point as radius.

AXIOMS.

I.

Things that are equal to the same thing are equal to one another.

II.

If equals, or the same thing, be added to equals, the sums are equal.

III.

If equals, or the same thing, be taken from equals, and if equals be taken from the same thing, the remainders are equal.

IV.

If equals, or the same thing, be added to unequals, the sums are unequal in the same kind of inequality.

V.

If equals, or the same thing, be taken from unequals, the remainders are unequal in the same kind of inequality.

VI.

Things that are double of the same thing, or of equals, are equal to one another.

VII.

Things that are halves of the same thing, or of equals, are equal to one another.

VIII.

Magnitudes that coincide with one another, i.e. exactly fill up the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot enclose a space.

XI.

All right angles are equal to one another.

XII.

If a straight line cut two straight lines, so as to make the two interior angles on the same side of it, when taken together, less than two right angles, then those two straight lines, being continually produced, shall at length meet on that side of the cutting line on which are the angles that are together less than two right angles.

Obs. 1. This axiom will be made clearer by illustration.

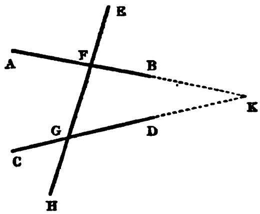


Fig. 1.

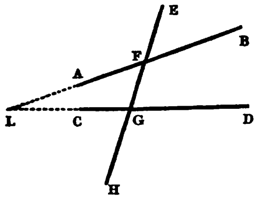


Fig. 2.

(1) Let the straight line EFGH (Fig. 1) cut the two straight lines AB, CD in the points F, G, so as to make the two interior angles BFG, FGD on the same side of EH (viz. that side towards B, D), when taken together, less than two right angles; then the 12th Axiom asserts that the two straight lines AB, CD, being continually produced (as represented by the dotted lines) shall at length meet in some point K upon the side of EH towards B, D, that being the side on which the angles BFG, FGD are, which are together less than two right angles.

(2) Let the straight line EFGH (Fig. 2) cut the two straight lines AB, CD in the points F, G, so as to make the two interior angles AFG, FGC on the same side of EH (viz. that side towards A, C), when taken together, less than two right angles; then the 12th

Axiom asserts that the two straight lines BA, DC, being continually produced (as represented by the dotted lines) shall at length meet in some point L on the side of EH towards A, C, that being the side on which the angles AFG, FGC are that are together less than two right angles.

OBS. 2. When, as in the figures in Obs. 1, a straight line EFGH cuts two other straight lines AB, CD in F, G, it makes with them eight angles, four on one side of EF, viz. AFE, AFG, FGC, CGH, and four on the other side of EF, viz. EFB, BFG, FGD, DGH.

Of these eight angles :—

(1) the angles AFE, EFB, CGH, HGD, are called exterior angles;

(2) the angles AFG, FGC, DGF, GFB, are called interior angles;

(3) any angle at F is said to be opposite to any angle at G;

(4) the interior angles which are opposite to one another, and on different sides of the cutting line, are called alternate interior, or alternate angles: thus AFG, FGD are alternate angles; BFG, FGC are alternate angles.

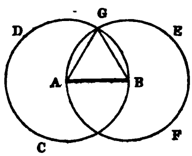
These remarks will render clearer the enunciations of Prop^s. 27, 28, 29.

PROPOSITIONS.

PROP. I. PROBLEM.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given finite straight line. It is required to describe an equilateral triangle on AB.



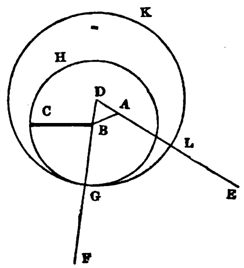
With one of the extremities A of AB as centre, and with AB as radius, describe (Post. 3) the circle BCD; with the other extremity B of AB as centre, and BA as radius, describe the circle AEF; and from the point G where these circles cut one another, draw (Post. 1) the straight lines GA, GB to the points A, B. Then ABG shall be the equilateral triangle required.

Because A is the centre of the circle BCD, AB is equal to AG by def^n (Def. 15); and because B is the centre of the circle AEF, BG is equal to BA for the same reason. Hence AG, BG are each of them equal to AB; and things that are equal to the same thing are equal to one another (Ax. 1): therefore AG is equal to BG, and the three straight lines AB, BG, GA, are all equal. Hence the triangle GAB is equilateral (Def. 24), and it has been described on the given straight line AB. Which was to be done.

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line. It is required to draw from A a straight line equal to BC.



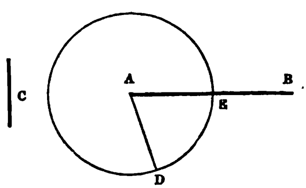
Join A with one of the extremities, as B, of the straight line BC (Post. 1); on AB describe the equilateral triangle ABD (i. 1); and produce the straight lines DA, DB to E, F (Post. 2). With centre B and radius BC describe (Post. 3) the circle CGH, cutting DF in G; and with centre D and radius DG describe the circle GKL, cutting DE in L. Then AL shall be equal to BC.

Because B is the centre of the circle CGH, BC is equal to BG by def^n; and because D is the centre of the circle GKL, DL is equal to DG for the same reason. Now DA is equal to DB, since they are sides of the equilateral triangle DAB; from the equals DL, DG, take away the equals DA, DB: then the remainders are equal (Ax. 3), or AL is equal to BG. But BC also is equal to BG; and things that are equal to the same thing are equal to one another (Ax. 1); therefore AL is equal to BC. Hence from the given point A a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

PROP. III. PROB.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.



From A, that extremity of AB to which the part to be cut off is required to be adjacent, draw (i. 2) the straight line AD equal to C; and with centre A and radius AD describe (Post. 3) the circle DEF, cutting AB in E. Then AE shall be equal to C.

Because A is the centre of the circle DEF, AE is equal to AD by defⁿ (Def. 15). But by construction the straight line C is equal to AD; and things that are equal to the same thing are equal to one another (Ax. 3): therefore AE is equal to C. Hence from AB the greater of the two straight lines AB and C, a part AE has been cut off equal to C the less. Which was to be done.

PROP. IV. THEOREM.

If two triangles have

(1) two sides of the one respectively equal to two sides of the other;

(2) the angle included by the two sides of the one equal to the angle included by the two sides of the other:

then these triangles shall be equal in every respect; i. e.

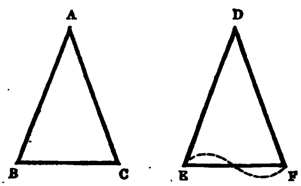
(1) the base or third side of the one shall be equal to the base or third side of the other;

(2) the remaining angles of the one shall be respectively equal to the remaining angles of the other, those angles being equal in each to which the equal sides are opposite;

(3) the triangles shall be equal.

Let ABC, DEF be two triangles, which have

(1) the two sides BA, AC of the one respectively equal to the two sides ED, DF of the other, viz. BA to ED, and AC to DF;



(2) the angle BAC included by the two sides BA, AC of the one equal to the angle EDF included by the two sides ED, DF of the other.

Then these triangles shall be equal in every respect: i. e.

(1) the base BC shall be equal to the base EF;

(2) the remaining angles ABC, ACB shall be equal to the remaining angles DEF, DFE respectively: viz.

ABC, DEF, to which the equal sides AC, DF are opposite, shall be equal; and ACB, DFE, to which the equal sides AB, DE are opposite, shall be equal;

(3) the triangle ABC shall be equal to the triangle DEF.

Let the triangle ABC be applied to the triangle DEF, so that the point A may coincide with the point D, and the straight line AB may fall on the straight line DE, the triangle ABC falling on the same side of DE as the triangle DEF.

Then the point A coinciding with the point D, and the straight line AB falling on DE by const^{n}, the point B shall coincide with the point E, because AB is equal to DE by hypothesis:

Again, the straight line AB coinciding with DE, the straight line AC shall fall on DF, because the angle BAC is equal to the angle EDF by hyp^{s}; and the triangles fall by const^{n} on the same side of DE: hence also the point C coincides with the point F, because AC is equal to DF by hyp^{s}:

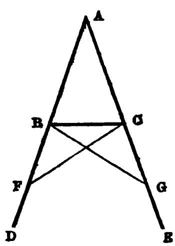
But the point B was shewn to coincide with the point E; hence the point B coinciding with the point E, and the point C coinciding with the point F, the straight line BC must coincide with the straight line EF; because if it did not, it would take some other as position in the figure, and there would be two straight lines inclosing a space, which is impossible (Ax. 10). Hence the straight line BC coincides with the straight line EF; and magnitudes which coincide are equal (Ax. 8): therefore the base BC is equal to the base EF. Also the whole triangle ABC coincides with the whole triangle DEF, and the remaining angles of the one coincide with the remaining angles of the other: therefore, for the same reason as before, the angles ABC, ACB are respectively equal to the angles DEF, DFE, and the triangle ABC is equal to the triangle DEF. Hence the two triangles have been shewn to be equal to one another in every respect as was enunciated. Which was to be proved.

PROP. V. THEOR.

If a triangle be isosceles: then

(1) the angles at the base shall be equal;

(2) if the equal sides be produced, the angles on the other side of the base shall be equal.



Let the triangle ABC be isosceles, having the side AB equal to the side AC. Then

(1) the angle ABC shall be equal to the angle ACB;

(2) if the equal sides AB, AC be produced to D and E, the angles CBD, BCE on the other side of the base BC shall be equal.

In BD take any point F; from AE cut off (i. 3) AG equal to AF; and join CF, BG.

Because AF is equal to AG by const^{n}, and AC to AB by hyp^{s}, and the angle at A is common to the two triangles FAC, GAB; therefore these two triangles have the two sides FA, AC respectively equal to the two sides GA, AB, and the included angle FAC equal to the included angle GAB. Therefore they are equal in every respect (i. 4); and hence the base FC is equal to the base GB, and the remaining angles ACF, AGB respectively equal to the remaining angles ABG, AFC, viz. ACF to ABG, and AFC to AGB:

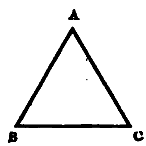
Again, AF is equal to AG, and AB is equal to AC; hence, taking away equals from equals, the remainder BF is equal (Ax. 3) to the remainder CG. Also, FC was shewn to be equal to CB, and the angle AFC was shewn to be equal to the angle AGB; therefore the two triangles BFC, CGB have the two sides BF, FC respectively equal to the two sides CG, GB, and the included angle BFC equal to the included angle CGB. Therefore they are equal (i. 4) in every respect; and hence the remaining angles FBC, FCB are respectively equal to the remaining angles GCB, GBC, viz. FBC to GCB, and FCB to GBC:

Now, it has been shewn that the angle ABG is equal to the angle ACF, and the angle GBC is equal to the

angle FCB: hence, taking away equals from equals, the remaining angle ABC is equal (Ax. 3) to the remaining angle ACB. And it was proved above that angle FBC is equal to angle GCB. Therefore (1) the angles ABC, ACB at the base BC are equal; (2) the angles CBD, BCE on the other side of the base BC are equal. Which was to be proved.

COR.—Every equilateral triangle shall also be equiangular.

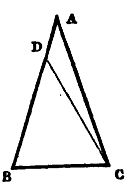
Let ABC be an equilateral triangle. Then it shall also be equiangular.



Since the triangle ABC is equilateral, the side AB is equal (Def. 24) to the side AC; and therefore by the propⁿ, the angles at the base BC, viz. ABC, ACB are equal. Again, since the triangle is equilateral, the side CA is equal to the side CB, and therefore by the propⁿ the angles CAB, ABC are equal. Hence each of the angles ACB, CAB is equal to the angle ABC; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle ACB is equal to the angle CAB. Hence the three angles ABC, BCA, CAB are all equal, that is, the triangle ABC is equiangular. Which was to be proved.

PROP. VI. THEOR.

If a triangle have two of its angles equal: then the sides which subtend or are opposite to the equal angles shall be equal.

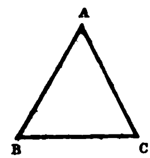


Let the triangle ABC have the angle ABC equal to the angle ACB. Then the side AC shall be equal to the side AB.

For if AB, AC be not equal, let them, if possible, be unequal, and let AB be the one which is greater than the other, AC. From BA cut off (i. 3) BD equal to AC; and join DC.

Because BD is equal to CA by const^n, BC common to the two triangles DBC, ACB, and the angles ABC, ACB equal by hyp^s; therefore these two triangles have the two sides DB, BC respectively equal to the two sides AC, CB, and the included angle DBC equal to the included angle ACB. Therefore they are equal (i. 4) in every respect; and hence the triangle DBC is equal to the triangle ABC, that is, the part equal to the whole, which is impossible (Ax. 9). Therefore AB, AC are not unequal, that is, the side AC is equal to the side AB. Which was to be proved.

COR.—Every equiangular triangle shall also be equilateral.



Let ABC be an equiangular triangle. Then it shall also be equilateral.

Since the triangle ABC is equiangular, the angle ABC is equal to the angle ACB; and therefore by the prop^n the sides opposite to them, viz. AC, AB, are equal. Again, since the triangle is equiangular, the angle ABC is equal to the angle CAB; and therefore the sides CA, CB are equal. Hence AB, BC are each of them equal to AC; and things that are equal to the same thing are equal to one another (Ax. 1): therefore AB is equal to BC. Hence the three sides AB, BC, CA are all equal, that is, the triangle ABC is equilateral (Def. 24). Which was to be proved.

PROP. VII.

On the same base and on the same side of it there cannot be two triangles which have their sides terminated in one extremity of the base equal, and likewise those terminated in the other extremity equal, not coinciding with one another.

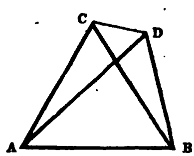
For, if there can be two such: let, if possible, on the same base AB, and on the same side of it, there be two triangles ACB, ADB, having their sides CA, DA termi-

nated in the extremity A of the base AB, equal, and likewise their sides CB, DB terminated in the other extremity B, equal, which two triangles do not coincide with one another.

There are three cases, according as the vertex (i. e. the angular point opposite to the base) of each of the triangles is without the other triangle; or the vertex D of one of them falls within the other, ACB; or the vertex D of one of them falls on a side CB of the other.

I. Let the vertex of each of the triangles fall without the other triangle.

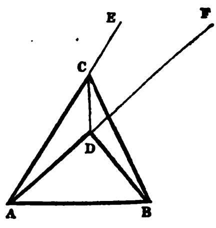
Join CD.



Because AC is equal to AD by hyp^{s}, the angle ACD is equal (i. 5) to the angle ADC. But the angle ACD is greater than the angle BCD (Ax. 9); therefore the angle ADC is greater also than the angle BCD: by much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to DB by hyp^{s}, the angle BDC is equal to the angle BCD; but it has been shewn to be greater than it: which is impossible.

II. Let the vertex of one of the triangles, as D, be within the other triangle, ACB.

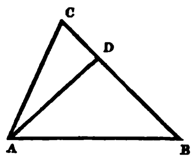
Join CD; and produce AC, AD, to E and F.



Because AC is equal to AD by hyp^{s}, and the equal sides AC, AD of the triangle ACD are produced to E, F: therefore the angles ACD, ADC on the other side of the base CD are equal (i. 5). But the angle ECD is greater (Ax. 9) than the angle BCD; therefore the angle FDC is greater also than the angle BDC: by much more then is the angle BDC greater than the angle BCD. Again, because BC is equal to BD by hyp^{s}, the angle BDC is equal (i. 5) to the angle BCD;

but it has been shewn to be greater than it: which is impossible.

III. Let the vertex of one of the triangles, as D, be on a side BC of the other.



Because the whole is greater than its part (Ax. 9), BC is greater than BD; but BC is equal to BD by hyp^s: which is impossible.

Hence in every case it has been shewn impossible for there to be on the same base AB, and on the same side of it, two triangles ACB, ADB, having their sides AC, AD terminated in one extremity A of the base equal, and likewise their sides BC, BD terminated in the other extremity B equal, which do not coincide with one another. Which was to be proved.

PROP. VIII. THEOR.

If two triangles have the three sides of the one respectively equal to the three sides of the other: then these triangles shall be equal in every respect, i. e.

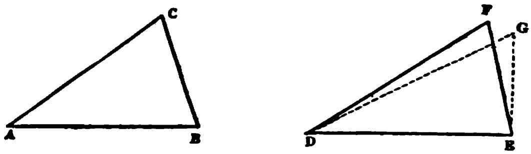
(1) the three angles of the one shall be respectively equal to the three angles of the other, those being the equal angles to which the equal sides are opposite;

(2) the triangles shall be equal.

Let ABC, DEF be two triangles, having the three sides AB, BC, CA of the one respectively equal to the three sides DE, EF, FD of the other, viz. AB to DE, BC to EF, CA to FD. Then these triangles shall be equal in every respect, viz.

(1) the three angles BCA, CAB, ABC, shall be respectively equal to the three angles EFD, FDE, DEF, viz. BCA shall be equal to EFD, to which the equal sides AB, DE are opposite; CAB to FDE, to which the equal sides CB, FE are opposite; ABC to DEF, to which the equal sides AC, DF are opposite;

(2) the triangle ABC shall be equal to the triangle DEF.



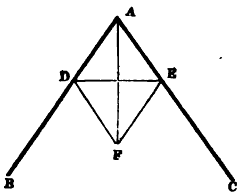
Let the triangle ABC be applied to the triangle DEF, so that the point A may coincide with the point D, and the straight line AB may fall on DE; the triangle ABC falling on the same side of DE as the triangle DEF.

Then the point A coinciding with the point D, and the straight line AB falling on the straight line DE, the point B shall coincide with the point E, because AB is equal to DE by hyp[s]; therefore the straight line AB coinciding with the straight line DE, and the triangles falling on the same side of DE, the sides AC, CB must coincide with the sides DF, FE: because, if AC, CB did not coincide with DF, FE, they would take some other position, as DG, GE, and on the same base DE and on the same side of it, there would be two triangles DGE, DFE, having their sides DG, DF terminated in one extremity D of the base equal, and likewise their sides GE, FE terminated in the other extremity E equal, which do not coincide with one another: which is impossible (i. 7). Therefore AC, CB coincide with DF, FE, and the triangle ABC coincides with the triangle DEF. Hence the three angles BCA, CAB, ABC coincide with, and are therefore equal (Ax. 8) to the three angles EFD, FDE, DEF respectively: and the triangle ABC coincides with, and is therefore equal to, DEF. Which was to be proved.

PROP. IX. PROB.

To bisect a given angle, i. e. to divide it into two equal angles.

Let BAC be the given angle. It is required to bisect the angle ABC, i. e. to divide it into two equal angles.

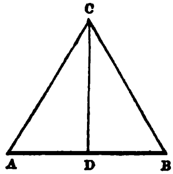


In AB take any point D; and from AC cut off (i. 3) AE equal to AD, join DE; on the side of DE opposite to A describe (i. 1) the equilateral triangle DFE; and join AF. Then the angle BAC shall be bisected by the straight line AF.

Because AD is equal to AE by const^n, and DF is equal to EF, since they are sides of the equilateral triangle DEF, and AF is common to the two triangles ADF, AEF; therefore these two triangles have the three sides AD, DF, FA of the one respectively equal to the three sides AE, EF, FA of the other. Therefore they are equal in every respect (i. 8); and hence the angle DAF is equal to the angle EAF. Hence the given angle BAC is bisected by the straight line AF. Which was to be done.

PROP. X. PROB.

To bisect a given finite straight line, i. e. to divide it into two equal parts.



Let AB be the given finite straight line. It is required to bisect AB, i. e. to divide it into two equal parts.

On AB describe (i. 1) the equilateral triangle ABC; bisect (i. 9) the angle ACB by the straight line CD, cutting AB in D. Then AB shall be bisected in the point D.

Because AC is equal to CB, since they are sides of the equilateral triangle ABC, CD common to the two triangles ACD, BCD, and the angle ACD equal to the angle BCD by const^n; therefore these two triangles have the two sides AC, CD respectively equal to the two sides BC, CD, and the included angle ACD equal to the included angle BCD. Therefore they are equal in every respect (i. 4); and hence the base AD is equal to the base DB. Hence the straight line AB is divided into two equal parts, or bisected in the point D. Which was to be done.

PROP. XI. PROB.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C the given point in it. It is required to draw from the point C a straight line at right angles to AB.

The given point C either lies between the extremities A, B, of AB (Fig. 1), or coincides with one of them, as B (Fig. 2).

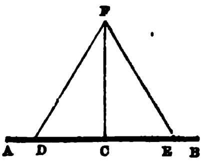


Fig. 1.

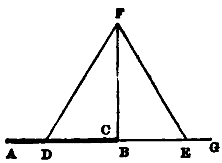
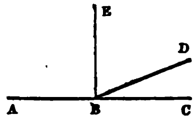


Fig. 2.

If C lie between A and B, in AC take any point D, and from CB, or CB produced if necessary, cut off (i. 3) CE equal to CD; or if C coincide with one extremity, as B, of AB, in AB take any point D, produce AB to G, and from BG or CG cut off (i. 3) CE equal to CD. On DE in both cases describe (i. 1) the equilateral triangle DFE; and join CF. Then CF shall be at right angles to AB.

Because FD is equal to FE, since they are sides of the equilateral triangle FDE; DC equal to CE by constⁿ; and FC common to the two triangles FDC, FEC: therefore these two triangles have the three sides FD, DC, CF respectively equal to the three sides FE, EC, CF. Therefore they are equal in every respect (i. 8); and hence the angle DCF is equal to the angle ECF. Thus the straight line FC standing on the straight line DE makes with it the adjacent angles DCF, ECF equal to one another: therefore by the defⁿ of a right angle (Def. 10), each of these angles is a right angle. Hence from the given point C in the given straight line AB, the straight line FC has been drawn at right angles to AB. Which was to be done.

COR.—Two straight lines cannot have a part common to both, or a common segment.



For if they can; let, if possible, the two straight lines ABC, ABD have the part AB common to both, or the common segment AB.

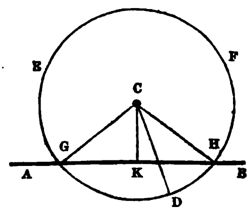
From B draw by the propⁿ BE at right angles to BA.

By the defⁿ of a right angle, since ABC is a straight line, and BE at right angles to it, the angle CBE is equal to the angle EBA; for the same reason, since ABD is a straight line, and BE at right angles to it, the angle DBE is equal to the angle EBA; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle CBE is equal to the angle DBE, that is, the whole equal to the part, which is impossible (Ax. 9). Hence two straight lines cannot have a common segment. Which was to be proved.

PROP. XII. PROB.

To draw a straight line perpendicular to a given straight line of indefinite length, from a given point without it.

Let AB be the given straight line of indefinite length, i. e. which may be produced to any length both ways, and let C be the given point without it. It is required from C to draw a straight line perpendicular to AB.



On the side of AB opposite to C take any point D; join CD; and with centre C and radius CD describe the circle EDF. Let G, H be the points where this circle cuts AB, or AB produced if necessary; bisect (i. 10) GH in K, and join CK. Then CK shall be perpendicular to AB.

Join CG and CH.

Because CG is equal to CH by the def^{n} of a circle, GK to KH by const^{n}; and CK common to the two triangles CGK, CHK; therefore these two triangles have the three sides CG, GK, KC respectively equal to the three sides CH, HK, KC. Therefore they are equal in every respect (i. 8); and hence the angle CKG is equal to the angle CKH. That is, the straight line CK standing on the straight line GH, makes with it the adjacent angles CKG, CKH equal to one another; therefore by the def^{n} of a right angle, each of these angles is a right angle, and each of the straight lines CK, GH is perpendicular to the other. Whence from the given point C a straight line CK has been drawn perpendicular to the given straight line AB. Which was to be done.

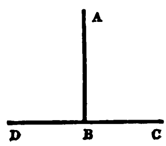
PROP. XIII. THEOR.

The adjacent angles which one straight line makes with another on the same side of it shall either be two right angles, or be together equal to two right angles.

Let the straight line AB make with the straight line CD on the same side of it the adjacent angles ABC, ABD. Then these angles shall either be two right angles, or be together equal to two right angles.

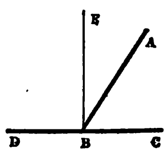
The angles ABC, ABD are either equal to one another or they are not.

I. Let them be equal.



Then since the angle ABC is equal to the angle ABD, the straight line AB standing on the straight line CD makes with it the adjacent angles ABC, ABD equal to one another: therefore by def^{n} of a right angle each of these angles is a right angle. Hence in this case the angles ABC, ABD are two right angles.

II. Let them not be equal.



From B draw (i. 11) BE at right angles to CD. By const^{n} the angles EBC, EBD are two right angles. And because the angle EBC is equal to the two angles EBA, ABC

together; to each of these equals add the angle DBE: therefore the angles EBC, EBD are equal (Ax. 2) to the three angles DBE, EBA, ABC:

Again, because the angle DBA is equal to the angles DBE, EBA; to each of these equals add the angle ABC: therefore the angles DBA, ABC are equal (Ax. 3) to the three angles DBE, EBA, ABC:

But it was shewn that the angles EBC, EBD are equal to the same three angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles ABC, ABD are equal to the angles EBC, EBD. Now, EBC, EBD are two right angles: therefore in this case the angles ABC, ABD are together equal to two right angles.

Hence the angles ABC, ABD are either two right angles, or are together equal to two right angles. Which was to be proved.

PROP. XIV. THEOR.

If at a point in a straight line, two other straight lines, on the opposite sides of it, make with it the adjacent angles together equal to two right angles: then these two straight lines shall be in one straight line.

At the point B in the straight line AB let the two straight lines BC, BD, on the opposite sides of AB, make with AB the adjacent angles ABC, ABD together equal to two right angles. Then BC, BD shall be in one straight line.

For if they are not, some other straight line than BD through B will be in the same straight line with CB: let, if possible, BE be in the same straight line with CB, BE either falling within the angle ABD (Fig. 1) or without it (Fig. 2).

Because AB makes with the straight line CBE on one side of it the adjacent angles ABC, ABE; these an-

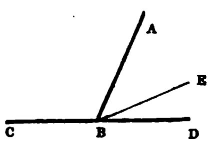


Fig. 1.

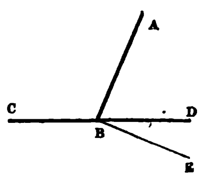


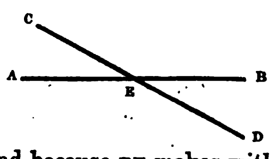
Fig. 2.

gles are equal (i. 13) to two right angles. But the angles ABC, ABD are likewise equal to two right angles by hyp[s]; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles ABC, ABE are equal to the angles ABC, ABD. From each of these equals take away the common angle ABC; then the remaining angle ABE is equal (Ax. 3) to the remaining angle ABD; that is, the part equal to the whole (Fig. 1), or the whole equal to the part (Fig. 2): which is impossible (Ax. 9). Therefore BE is not in the same straight line with BC. And in like manner it may be proved that no other straight line through B on the other side of AB can be in the same straight line with CB but BD. Therefore BD is in the same straight line with CB; or CB, DB are in one straight line. Which was to be proved.

PROP. XV. THEOR.

If two straight lines cut one another: then of the four angles made at the point of intersection (or vertical angles), those which are opposite shall be equal to one another.

Let the two straight lines AB, CD cut one another in E. Then the opposite vertical angles shall be equal, i. e. the angle AEC shall be equal to the angle BED, and the angle BEC to the angle AED.



I. The angle AEC shall be equal to the angle BED.

Because AE makes with CD on the same side of it, the adjacent angles AEC, AED; these angles are equal (i. 13) to two right angles. And because DE makes with AB on the same side of it the adjacent angles AED, BED: these angles are also equal to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles AEC, AED are equal to the angles AED, BED. From each of these equals take

away the common angle AED: then the remaining angle AEC is equal (Ax. 3) to the remaining angle BED.

II. The angle BEC shall be equal to the angle AED.

The proof of this case is exactly similar to that of the first case.

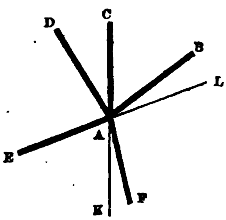
Hence the opposite vertical angles AEC, BED are equal, and the opposite vertical angles BEC, AED are equal. Which was to be proved.

COR. 1.—If two straight lines cut one another: then the four angles which they make at the point of intersection shall be together equal to four right angles.

Let AB, CD cut one another in E. Then the four angles AEC, AED, DEB, BEC shall be together equal to two right angles.

By the propⁿ it was proved that the angles AEC, AED are equal to two right angles, and that BEC, BED are equal to two right angles. Therefore adding equals to equals, the four angles AEC, AED, BEC, BED are together equal (Ax. 2) to four right angles. Which was to be proved.

COR. 2.—The angles made by any number of straight lines meeting in one point shall be together equal to four right angles.



Let AB, AC, AD, AE, AF be any number of straight lines meeting in one point A. Then the angles BAC, CAD, DAE, EAF, FAD shall be together equal to four right angles.

Produce any two of the straight lines as CA, EA, to K, L.

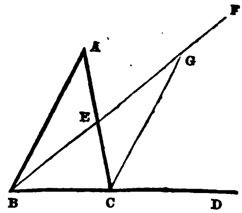
Then the angles which CK, EL make at A where they cut one another, i. e. the four angles, CAE, EAK, KAL, LAC are together equal to four right angles by Cor. 1. But it may be shewn as in the proof of the 13th Propⁿ that the angles CAE, EAK

are equal to the angles CAD, DAE, EAK, and that the angles KAL, LAC are equal to the angles KAF, FAL, LAB, BAC: therefore, adding equals to equals, the four angles CAE, EAK, KAL, LAC are equal to (Ax. 2) the angles CAD, DAE, EAF, FAB, BAC. But the same four angles were shewn to be equal to four right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles CAD, DAE, EAF, FAB, BAC are together equal to four right angles. Which was to be proved.

PROP. XVI. THEOR.

If one side of a triangle be produced: then the exterior angle shall be greater than either of the interior and opposite angles.

Let ABC be a triangle, and let one of its sides BC be produced to D. Then the exterior angle ACD shall be greater than either of the two interior and opposite angles CAB, ABC.

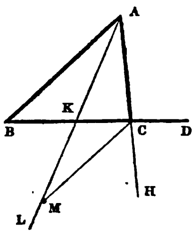


I. It shall be greater than the interior and opposite angle BAC.

Bisect (i. 10) AC in E, and join BE. Produce BE to F; from EF cut off (i. 3) EG equal to EB; and join CG.

Because by constn AE is equal to EC, and BE to EG, and the angle AEB is equal to the angle CEG, since they are opposite vertical angles (i. 15); therefore the two triangles AEB, CEG have the two sides AE, EB respectively equal to the two sides CE, EG, and the included angle AEB equal to the included angle CEG. Therefore these two triangles are equal in every respect (i. 4); and hence the angle BAE is equal to the angle ECG. But the angle ECD is greater than the angle ECG (Ax. 9): therefore the angle ECD or ACD is greater than the angle BAE or BAC.

II. It shall be greater than the interior and opposite angle ABC.



Produce AC to H; bisect (I. 10) BC in K, and join AK. Produce AK to L; from KL cut off (I. 3) KM equal to KA; and join CM.

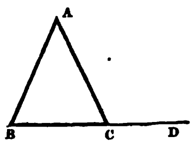
By constr BK is equal to KC, and AK to KM, and the angle AKB is equal to the angle MKC, since they are opposite vertical angles (I. 15): therefore the two triangles AKB, MKC have the two sides AK, KB respectively equal to the two sides MK, KC, and the included angle AKB equal to the included angle MKC. Therefore these two triangles are equal in every respect (I. 4); and hence the angle ABK is equal to the angle MCK. But the angle BCH is greater than the angle MCK (Ax. 9); therefore the angle BCH is greater than the angle ABK or ABC; and the angle ACD is equal to the angle BCH, since they are opposite vertical angles: therefore the angle ACD is greater than the angle ABC.

Hence the exterior angle ACD is greater than either of the two interior and opposite angles CAB, ABC. Which was to be proved.

PROP. XVII. THEOR.

Any two angles of a triangle shall be together less than two right angles.

Let ABC be any triangle. Then any two of its angles shall be together less than two right angles.



Produce one of its sides BC to D.

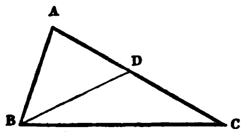
Because the side BC of the triangle ABC is produced to D, the exterior angle ACD is greater (I. 16) than the interior and opposite angle ABC. To each of these unequals add the angle ACB; therefore the angles ACD, ACB are greater (Ax. 4) than the angles ABC, ACB. But since AC makes with BD on one side of it the adjacent angles ACB, ACB, these angles are equal (I. 13) to two right angles: therefore the angles ABC, ACB are less

than two right angles. In like manner it may be shewn that the two angles BAC, ACB, and that the two angles CAB, ABC are less than two right angles. Hence any two angles of the triangle ABC are together less than two right angles. Which was to be proved.

PROP. XVIII. THEOR.

The greater side of any triangle shall be opposite to the greater angle; i. e. if one side of a triangle be greater than another, then the angle opposite to the greater side shall be greater than the angle opposite to the less side.

Let ABC be a triangle, of which the side AC is greater than the side AB. Then the angle ABC shall be greater than the angle BCA.



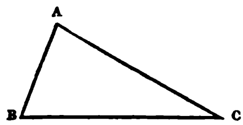
From AC, which is by hyp^s greater than AB, cut off (i. 3) AD equal to AB; and join BD.

Because the side CD of the triangle BCD is produced to A, the exterior angle BDA is greater (i. 16) than the interior and opposite angle DCB; and because AD is equal to AB by const^n, the angle BDA is equal (i. 5) to the angle ABD; therefore the angle ABD is likewise greater than the angle ACB. By much more then is the angle ABC greater than the angle ACB. Which was to be proved.

PROP. XIX. THEOR.

The greater angle of any triangle shall be subtended by the greater side; i. e. if one angle of a triangle be greater than another, then the side opposite to the greater angle shall be greater than the side opposite to the less angle.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA. Then the side AC shall be greater than the side AB.

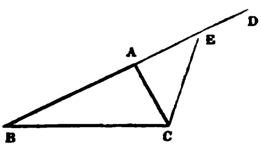


For if it be not greater, AC must either be equal to AB or less than AB. It is not equal; because then the angle ABC would be equal (I. 5) to the angle ACB; but it is not: therefore AC is not equal to AB. Neither is it less; because then the angle ABC would be less than the angle ACB, since the greater side of any triangle is opposite to the greater angle (I. 18); but it is not: therefore AC is not less than AB. Hence since the side AC has been shewn to be neither equal to the side AB, nor less than it, it must be greater than the side AB. Which was to be proved.

PROP. XX. THEOR.

Any two sides of a triangle shall be together greater than the third side.

Let ABC be a triangle. Then any two of its sides together shall be greater than the third side.



Produce one of the sides, BA, to D; from AD cut off (I. 3) AE equal to AC; and join CE.

Because AE is equal to AC by const^n, the angle AEC is equal (I. 5) to the angle ACE. But the angle BCE is greater (Ax. 9) than the angle ACE; therefore the angle BCE is greater than the angle AEC, or BEC: and the greater angle of any triangle is subtended by the greater side (I. 19): therefore EB is greater than BC. Now EB is equal to BA and AC together, since AC is equal to AE by const^n: therefore the sides BA, AC are greater than the side BC. In like manner it may be shewn that the sides AB, BC are greater than the side AC: and that the sides BC, CA are greater than AB. Hence any two sides of the triangle ABC are together greater than the third. Which was to be proved.

PROP. XXI. THEOR.

If from the extremities of any side of a triangle there be drawn two straight lines to a point within the triangle: then these two straight lines shall together be less than the other two sides of the triangle, and shall contain a greater angle.

Let the two straight lines BD, CD be drawn from the extremities B, C of the side BC of the triangle ABC to the point D within it. Then:—

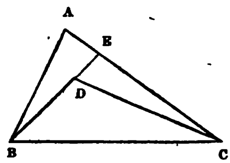
I. BD and DC shall be together less than BA and AC.

Produce BD to meet CA in E.

The two sides BA, AE of the triangle BAE are greater (i. 20) than the third BE; to each of these unequals add EC: then BA, AC are greater (Ax. 4) than BE, EC. Again, the two sides CE, ED of the triangle CED are greater than the third CD; to each of these unequals add DB: then CE, EB are greater than CD, DB. But it has been shewn that BA, AC are greater than BE, EC: by much more then are BA, AC greater than BD, DC. Which was to be proved.

II. The angle BDC shall be greater than the angle BAC.

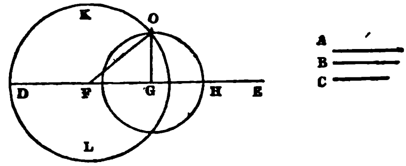
Construct as before.



Because the side ED of the triangle CDE is produced to B, the exterior angle BDC is greater (i. 16) than the interior and opposite angle CED. And because the side AE of the triangle BAE is produced to C, the exterior angle BEC is greater than the interior and opposite angle BAE. But it has been shewn that the angle BDC is greater than the angle BEC: by much more then is the angle BDC greater than the angle BAC. Which was to be proved.

PROP. XXII. PROB.

To make a triangle of which the sides shall be equal to three given straight lines, any two of which are together greater (i. 20) **than the third.**



Let A, B, C be the three given straight lines, of which any two are together greater than the third, viz. A and B together greater than C, C and A together greater than B, and B and C together greater than A. It is required to make a triangle of which the sides shall be equal to A, B, C respectively.

Take a straight line DE terminated at D, but produced indefinitely towards E; from DE cut off (i. 3) DF equal to A, from FE cut off FG equal to B, and from GE cut off GH equal to C. With centre F and radius FD describe the circle DKL; with centre G and radius GH describe the circle HMN: and from O where these circles cut one another, draw to F, G the straight lines OF, OG. Then the triangle OFG shall be the triangle required.

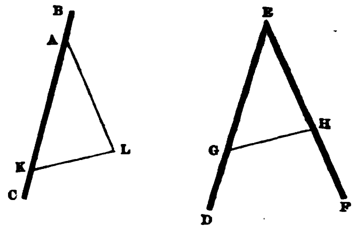
Because F is the centre of the circle DOL, and G that of OMN, FO is equal to FD, and GO to GH by def^{n}. But by const^{n} DF is equal to A, and GH equal to C; and things that are equal to the same thing are equal to one another (Ax. 1): therefore FO is equal to A, and GO to C; and FG is equal to B by const^{n}. Hence a triangle OFG has been made having its three sides OF, FG, GO respectively equal to A, B, C. Which was to be done.

PROP. XXIII. PROB.

At a given point, in a given straight line, to make an angle equal to a given ngle.

Let A be the given point in the given straight line BC, and DEF the given angle. It is required to make an angle at A in BC, that shall be equal to the angle DEF.

In ED, EF take any two points G, H; join GH; and make (i. 22) the triangle AKL so that its sides shall be respectively equal to the three straight lines EG, GH, HE, any two of which, since they are sides of a triangle, are greater (i. 20) than the third, viz. so that AK shall be equal to EG, KL to GH, and LA to HE. Then the angle CAL shall be equal to the angle GEH.



By constⁿ the three sides AK, KL, LA of the triangle AKL are respectively equal to the three sides EG, GH, HE of the triangle EGH. Therefore these two triangles are equal in every respect (i. 8); and hence the angle KAL is equal to the angle GEH. Therefore at the given point A in the given straight line BC, the angle CAL has been made equal to the given angle DEF. Which was to be done.

PROP. XXIV. THEOR.

If two triangles have

(1) two sides of the one respectively equal to two sides of the other;

(2) the angle included by the two sides of the one greater than the angle included by the two sides of the other:

then the base of the triangle which has the greater included angle shall be greater than the base of the other triangle.

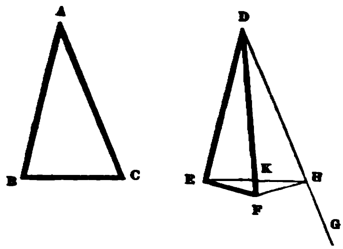
Let ABC, DEF be two triangles, which have

(1) the two sides BA, AC of the one respectively equal to the two sides ED, DF of the other, viz. BA to ED, and AC to DF;

(2) the angle BAC included by the two sides BA, AC of the one greater than the angle EDF included by the two sides ED, DF of the other.

Then the base BC of the triangle ABC which has the greater included angle shall be greater than the base EF of the other triangle DEF.

Of the two sides DE, DF, which are either equal or of which one is greater than the other, let DE be that which is not greater than the other DF; and at the point D, in the straight line DE, and on the same side of DE as DF is,



make the (i. 23) angle EDG equal to the angle BAC. From DG cut off (i. 3) DH equal to DF or AC; join FH, HE; and since EH must cut either DF, or DF produced, let K be the point where it cuts DF or DF produced, if necessary.

Because DE is supposed not greater than DH, and the greater side of a triangle is opposite to the greater angle (i. 18): therefore the angle DHE is not greater than the angle DEH. And because the side EK of the triangle DEK is produced to H, the exterior angle DKH is greater (i. 16) than the interior and opposite angle DEK. Therefore the angle DKH is greater than the angle DHK; and the greater angle of a triangle is subtended by the greater side (i. 19): therefore DH is greater than DK. But DF is equal to DH or AC by const^{n}: therefore DF is greater than DK. Hence K where EH cuts DF or DF produced if necessary, is a point in DF; and F falls on the opposite side of EG to which D does:

Because AB is equal to DE by hyp^{s}, AC to DH by const^{n}, and the angle BAC equal to the angle EDH for the same reason: therefore the two triangles BAC, EDH have the two sides BA, AC equal to the two sides ED, DH respectively, and the included angle BAC equal to the included angle EDH. Therefore these two triangles are equal in every respect (i. 4); and hence the base BC is equal to the base EH. And because F was shewn to fall on the opposite side of EH to which D does,

the angle EHF is a part of the angle DHF; therefore the angle FHD is greater (Ax. 9) than the angle EHF. But since DF is equal to DH by const^{n}, the angle DFH is equal (i. 5) to the angle DHF: therefore the angle DFH is greater than the angle EHF. By much more then is the angle EFH greater than the angle EHF; and the greater angle of a triangle is subtended by the greater side (i. 19): therefore EH is greater than EF. Now EH has been shewn to be equal to BC; hence BC is greater than EF, that is, the base BC of the triangle ABC, which has the greater included angle BAC, is greater than the base EF of the other triangle DEF. Which was to be proved.

PROP. XXV. THEOR.

If two triangles have

(1) two sides of the one respectively equal to two sides of the other;

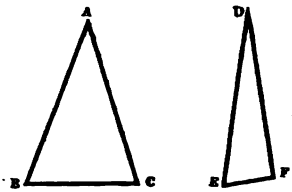
(2) the base of the one greater than the base of the other:

then, the angle included by the two sides of the one that has the greater base shall be greater than the angle included by the two sides of the other.

Let ABC, DEF be two triangles, which have

(1) the two sides BA, AC of the one respectively equal to the two sides ED, DF of the other; viz. BA to ED, and AC to DF;

(2) the base BC of the one greater than the base EF of the other.



Then the angle BAC included by BA, AC shall be greater than the angle EDF included by ED, DF.

For if it be not; the angle BAC must either be equal to the angle EDF, or less than the angle EDF. It is not equal: because then the two triangles ABC, DEF would

have the two sides BA, AC respectively equal to the two sides ED, DF, and the included angle BAC equal to the included angle EDF; and therefore they would be equal in every respect (I. 4), and hence the base BC would be equal to the base EF; but it is not: therefore the angle BAC is not equal to the angle EDF. Neither is it less: because then the two triangles ABC, DEF having their two sides equal as before, would have the included angle BAC less than the included angle EDF; and therefore the base BC would be less (I. 24) than the base EF; but it is not: therefore the angle BAC is not less than the angle EDF. Therefore the angle BAC, since it has been shewn to be neither equal to nor less than the angle EDF, must be greater than it. Which was to be proved.

PROP. XXVI. THEOR.

If two triangles have

(1) two angles of the one respectively equal to two angles of the other;

(2) one side of the one equal to one side of the other; the equal sides being either those adjacent to the two equal angles in both triangles, or those opposite to equal angles in each triangle:

then these two triangles shall be equal in every respect, i. e.

(1) the third angle of the one shall be equal to the third angle of the other;

(2) the remaining sides of the one shall be respectively equal to the remaining sides of the other, those sides being equal in each to which the equal angles are opposite, or which are adjacent to equal angles;

(3) the triangles shall be equal.

Let ABC, DEF be two triangles, which have the two angles ABC, ACB of the one respectively equal to the two angles DEF, DFE of the other, viz. ABC to DEF, and ACB to DFE; and:—

I. Let the sides BC, EF, those adjacent to the two equal angles, be equal; BC being adjacent to ABC, ACB, and EF

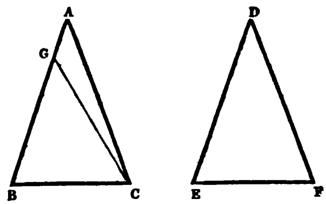
to DEF, DFE. Then these two triangles shall be equal in every respect, i. e.

(1) the third angle BAC shall be equal to the third angle EDF;

(2) the remaining sides BA, AC shall be respectively equal to the remaining sides ED, DF: viz. AB, DE which are opposite to the equal angles ACB, DFE shall be equal; and AC, DF which are opposite to the equal angles ABC, DEF shall be equal;

(3) the triangle ABC shall be equal to the triangle DEF.

For, if the sides AB, DE be not equal, let them, if possible, be unequal, and let AB be the one which is greater than the other DE. From BA cut off (i. 3) BG equal to DE; and join CG.



Then, because the angle ABC is equal to the angle DEF and BC to EF by hyp[s], and BG is equal to ED by const[n]; therefore the triangles GBC, DEF have the two sides GB, BC respectively equal to the two sides DE, EF, and the included angle GBC equal to the included angle DEF. Therefore these two triangles are equal in every respect (i. 4); and hence the angle GCB is equal to the angle DFE. But the angle ACB is equal to the angle DFE by hyp[s]; and things that are equal to the same thing are equal to one another (Ax. 1); therefore the angle GCB is equal to the angle ACB, that is, the part equal to the whole: which is impossible (Ax. 9). Therefore the sides AB, DE are not unequal, that is, they are equal. Also BC is equal to EF, and the angle ABC to the angle DEF; therefore the two triangles ABC, DEF have the two sides AB, BC respectively equal to the two sides DE, EF, and the included angle ABC equal to the included angle DEF. Therefore these two triangles are equal in every respect; and hence the side AC is equal to the side [1] angle BAC to the third angle EDF, and the

triangle ABC to the triangle DEF. Which was to be proved.

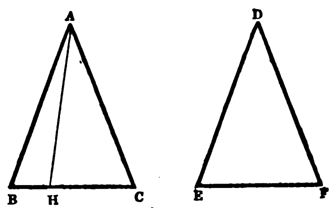
II. Let the sides AC, DF, those opposite to ABC, DEF, one of the two pairs of equal angles, be equal; AC being opposite to the angle ABC, and DF to the angle DEF. Then these two triangles shall be equal in every respect, i. e.

(1) the third angle BAC shall be equal to the third angle EDF;

(2) the remaining sides AB, BC shall be respectively equal to the remaining sides DE, EF; viz. AB, DE which are opposite to the equal angles ACB, DFE shall be equal; and BC, EF which are adjacent to the two equal angles in both triangles, shall be equal;

(3) the triangle ABC shall be equal to the triangle DEF.

For if the sides BC, EF be not equal, let them, if possible, be unequal; and let BC be the one which is greater than the other EF. From CB cut off (i. 3) CH equal to EF; and join AH.

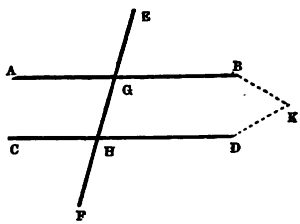


Then, because the angle ACB is equal to the angle DFE and AC to DF by hyp[s], and CH is equal to EF by const[n]; therefore the two triangles AHC, DEF have the two sides HC, CA respectively equal to the two sides EF, FD, and the included angle HCA equal to the included angle EFD. Therefore these two triangles are equal in every respect (i. 4); and hence the angle AHC is equal to the angle DEF. But the angle DEF is equal to the angle ABC by hyp[s]; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle AHC is equal to the angle ABC. But since the side BH of the triangle ABH is produced to C, the exterior angle AHC is greater (i. 16) than the interior and opposite angle ABH; and it has just been shewn to be equal to it: which is impossible. Therefore

the sides BC, EF are not unequal, that is, they are equal. Also AC is equal to DF, and the angle ACB to the angle DFE. Therefore the two triangles ABC, DEF have the two sides BC, CA respectively equal to the two sides EF, FD, and the included angle BCA equal to the included angle EFD. Therefore these two triangles are equal in every respect; and hence the side AB is equal to the side DE, the third angle BAC to the third angle EDF, and the triangle ABC to the triangle DEF. Which was to be proved.

PROP. XXVII. THEOR.

If a straight line cutting two other straight lines make either pair of alternate angles equal to one another: then these two straight lines shall be parallel.



Let EF, cutting the two straight lines AB, CD in G, H, make the pair of alternate angles AGH, GHD equal to one another. Then AB shall be parallel to CD.

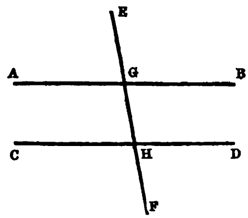
For if not: let, if possible AB, CD be not parallel. Then they being produced far enough must meet either towards B, D, or towards A, C. First let them be produced and meet towards B, D in K.

Then GHK is a triangle. And because the side KBG is produced to A, the exterior angle AGH is greater (i. 16) than the interior and opposite angle GHK; but by hyp[s] the angle AGH is also equal to the angle GHK: which is impossible. Therefore AB and CD being produced do not meet towards B, D. In like manner it may be shewn that they do not, being produced, meet towards A, C. Hence the straight lines in the same plane AB, CD being produced ever so far both ways do not meet: therefore by the def[n] (Def. 35) of parallel straight lines, AB is parallel to CD. And the same thing might have been shewn, if EF had made the other pair of alternate angles BGH, GHC

equal. Hence, if EF cutting the two straight lines AB, CD in G, H make either pair of alternate angles equal, AB, CD are parallel. Which was to be proved.

PROP. XXVIII. THEOR.

If a straight line cutting two other straight lines make one of the exterior angles equal to the interior and opposite angle on the same side of the cutting line: then these two straight lines shall be parallel. And if a straight line cutting two other straight lines make the two interior angles on the same side of the cutting line together equal to two right angles: then these two straight lines shall be parallel.



I. Let EF, cutting the two straight lines AB, CD in G, H, make one of the exterior angles, as EGB, equal to the interior and opposite angle on the same side of EF, viz. GHD. Then AB shall be parallel to CD.

Because the angle EGB is equal to the angle GHD by hyp[s], and the angle EGB is equal to the angle ACB, since they are opposite vertical angles (i. 15); and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle AGH is equal to the angle GHD. Hence EF, cutting the two straight lines AB, CD in G, H, makes the alternate angles AGH, GHD equal: therefore AB is parallel (i. 27) to CD. Similarly, if the exterior angle and interior and opposite angle on the same side, that AB makes equal, had been DHF, BGH; or CHF, AGH; or EGA, GHC; it might have been shewn that AB was parallel to CD. Which was to be proved.

II. Let EF, cutting the two straight lines AB, CD in G, H, make the two interior angles on the same side of EF, as BGH, GHD on the side of EF towards B, D together equal to two right angles. Then AB shall be parallel to CD.

Because GH makes with AB, on the same side of it, the adjacent angles AGH, BGH; these angles are equal (i. 13)

to two right angles. But the angles BGH, GHD are equal to two right angles by hyp[s]; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles AGH, BGH are equal to the angles BGH, GHD. From each of these equals take away the common angle BGH: then the remaining angle AGH is equal (Ax. 3) to the remaining angle GHD. Hence EF cutting the two straight lines AB, CD in G, H, makes the alternate angles AGH, GHD equal: therefore AB is parallel (i. 27) to CD. Similarly, if the interior angles on the same side of EF which AB makes equal to two right angles, had been AGH, GHC, on the side of EF towards A, C, it might have been shewn that AB is parallel to CD. Which was to be proved.

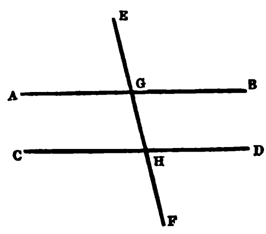
PROP. XXIX. THEOR.

If a straight line cut two parallel straight lines: then it shall make

(1) the alternate angles equal to one another;

(2) any one of the exterior angles equal to the interior and opposite angle on the same side of the cutting line;

(3) the two interior angles on the same side of the cutting line together equal to two right angles.



Let the straight line EF cut the two parallel straight lines AB, CD in G, H. Then:—

I. The alternate angles shall be equal; viz. AGH to GHD, and BGH to GHC.

For if AGH, GHD be not equal: let them, if possible, be unequal, and let AGH be the one which is greater than the other, GHD.

Since the angle AGH is greater than the angle GHD, to each of these unequals add the angle BGH; then the angles AGH, BGH are greater (Ax. 4) than the angles BGH, GHD. But because HG makes with AB, on the same side of it, the adjacent angles AGH, HGB, these angles

are equal (i. 13) to two right angles; and therefore the angles BGH, GHD are less than two right angles. Hence the straight line EF, cutting the two straight lines AB, CD in G, H, makes the two interior angles BGH, GHD, on the same side of EF, together less than two right angles: therefore by the axiom (Ax. 12) AB, CD, being continually produced, shall at length meet in some point on the side of EF towards B, D. But they can never meet, if produced ever so far, since they are parallel by hyp[s]: which is absurd. Therefore the angles AGH, GHD are not unequal, that is, the alternate angles AGH, GHD are equal. In like manner it may be shewn that the alternate angles BGH, GHC are equal. Which was to be proved.

II. Any one of the exterior angles, EGB, shall be equal to the interior and opposite angle on the same side of EF, viz. GHD.

The angle EGB is equal to the angle AGH, since they are opposite vertical angles (i. 15). And by Part I. of the prop[n] the angle GHD is equal to the angle AGH; and things that are equal to the same things are equal to one another (Ax. 1): therefore the angle EGB is equal to the angle GHD. In like manner it may be shewn that the angle DHF is equal to the angle HGB; the angle FHC to the angle HGA; and the angle EGA to the angle GHC. Hence any one of the exterior angles is equal to the interior and opposite angle on the same side of EF. Which was to be proved.

III. The two interior angles on the same side of EF shall be together equal to two right angles.

The angle EGB is equal to the angle GHD by Part II. of the prop[n]; to each of these equals add the angle BGH: then the angles EGB, BGH are equal (Ax. 2) to the angles BGH, GHD. But because BG makes with EH, on the same side of it, the adjacent angles BGE, BGH, these angles are equal (i. 13) to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles BGH, GHD, that is, the two interior angles on the side of EF towards B, D, are equal to two right angles. In like manner it may be shewn that the

interior angles AGH, GHC, on the side of EF towards A, C, are equal to two right angles. Which was to be proved.

PROP. XXX. THEOR.

Two straight lines that are each of them parallel to the same straight line shall be parallel to one another.

Let the two straight lines AB, CD be each of them parallel to the same straight line EF. Then AB shall be parallel to CD.

Draw any straight line GH, cutting AB in K, CD in L, and EF in M.

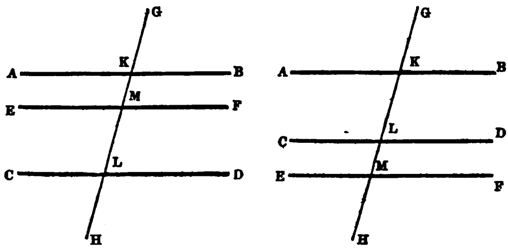


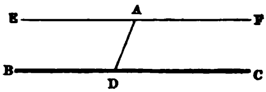
Fig. 1. Fig. 2.

The straight line EF lies either between AB and CD (Fig. 1), or without them, on the side of one, as CD (Fig. 2). Now, because GH cuts the parallels CD, EF in M, L; therefore in Fig. 1, the exterior angle KMF is equal (i. 29) to the interior and opposite angle on the same side, KLD, and in Fig. 2, the exterior angle KLD is equal to the interior and opposite angle LMF, on the same side; that is, in both figures the angle KMF is equal to the angle KLF. But because GH cuts the parallels AB, EF in K, M, therefore in both figures the alternate angles AKM, KMF are equal; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle AKM is equal to the angle KLD. Hence GH, cutting the two straight lines AB, CD in K, L, makes the alternate angles AKL, KLD equal to one another: therefore AB is parallel (i. 27) to CD. Which was to be proved.

PROP. XXXI. PROB.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point and BC the given straight line. It is required to draw through A a straight line parallel to BC.



In BC take any point D; and join AD. At the point A in the straight line AD, make (i. 23) the angle DAE equal to the angle ADC, and on the opposite side of AD to the angle ADC; and produce EA to F. Then EF shall be parallel to BC.

Because AD cutting the two straight lines EF, BC in A, D, makes the alternate angles EAD, ADC equal to one another; therefore EF is parallel (i. 27) to BC. Hence a straight line EF has been drawn through the given point A parallel to the given straight line BC. Which was to be done.

PROP. XXXII. THEOR.

In every triangle,

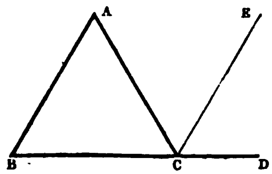
(1) if any side be produced, then the exterior angle shall be equal to the two interior and opposite angles;

(2) the three interior angles shall be together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D. Then

(1) the exterior angle ACD shall be equal to the two interior and opposite angles ABC, BAC;

(2) the three interior angles of the triangle, viz. ABC, BCA, CAB shall be together equal to two right angles.

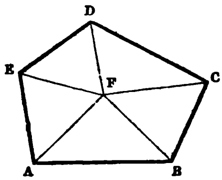


Through C draw (i. 31) the straight line CE parallel to BA.

Because AC cuts the parallels AB, CE in A, C, the alternate angles BAC, CAE are equal (i. 29). And because BD cuts the parallels AB, CE in B, C, the exterior angle ECD is equal to the interior and opposite angle on the same side ABC; but the angle ACE was shewn to be equal to the angle CAB: therefore adding equals to equals, the whole angle ACD is equal (Ax. 2) to the two angles ABC, CAB. To each of these equals add the angle ACB: then the angles ACD, ACB are equal to the three angles ABC, BCA, CAB. But because AC makes with BD on the same side of it the adjacent angles ACD, ACB, these angles are equal (i. 13) to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the three angles ABC, BCA, CAB are equal to two right angles. Whence (1) the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC; (2) the three interior angles of the triangle, viz. ABC, BCA, CAB are together equal to two right angles. Which was to be proved.

COR. 1.—All the interior angles of any polygon together with four right angles shall be equal to twice as many right angles as the polygon has sides.

Let ABCDE be any polygon. Then all the angles at A, B, C, D, E together with four right angles shall be equal to twice as many right angles as the figure ABCDE has sides.

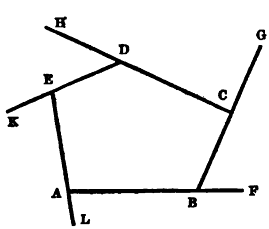


Within the figure take any point F; and draw the straight lines FA, FB, FC, FD, FE from F to each of the angular points of the figure.

There are as many triangles FAB, FBC, FCD, FDE, FEA having a common angular point F as the polygon ABCDE has sides; and the angles of each of these triangles by the propⁿ are equal to two right angles: therefore all the angles of the triangles are equal to twice as many right angles

as the polygon has sides. Again, all the angles of the triangles are equal to the angles at the bases of the triangles, viz. FAB, FBA, FBC, FCB, FCD, FDC, FDE, FED, FEA, FAE together with the angles at the common angular point F, viz. AFB, BFC, CFD, DFE, EFA. But the angles at the bases of the triangles, viz. FAB, FBA, &c. are together equal to the angles of the polygon, viz. ABC, BCD, CDE, DEA, EAB; and the angles at the common angular point F, since they are the angles made by any number of straight lines FA, FB, FC, FD, FE meeting in one point F, are equal (i. 15, Cor. 2) to four right angles: therefore the angles of the triangles are equal to all the angles of the polygon together with four right angles. Now it has been shewn that the angles of the triangles are equal to twice as many right angles as the polygon has sides; and things that are equal to the same thing are equal to one another (Ax. 1): therefore all the angles of the polygon ABCDE, together with four right angles are equal to twice as many right angles as the polygon has sides. Which was to be proved.

COR. 2.—If one of the two sides including each angle of any polygon be produced, so as to form as many exterior angles as the polygon has angular points: then all the exterior angles shall be together equal to four right angles.



Let ABCDE be any polygon; and let its sides AB, BC, CD, DE, EA be produced to F, G, H, K, L. Then all the exterior angles GBF, DCG, EDH, AEK, BAL shall be together equal to four right angles.

Because CB makes with AF on the same side of it the adjacent angles CBA, CBF, these two angles are equal (i. 13) to two right angles. In like manner it may be shewn that

each of the other interior angles with its corresponding exterior angle is equal to two right angles; therefore all the interior angles, viz. ABC, BCD, CDE, DEA, EAB together with all the exterior angles CBF, DCG, EDH, AEK, BAL are equal to twice as many right angles as the polygon has angular points, that is, as it has sides. But all the interior angles together with four right angles (by Cor. 1) are equal to twice as many right angles as the polygon has sides; and things that are equal to the same thing are equal to one another (Ax. 1): therefore all the interior angles together with all the exterior angles are equal to all the interior angles together with four right angles. From each of these equals take away all the interior angles; then the remaining exterior angles of the polygon ABCDE, viz. CBF, DCG, EDH, AEK, BAL are together equal (Ax. 3) to four right angles. Which was to be proved.

COR. A.—If two triangles have two angles of the one respectively equal to two angles of the other; then the third angle of the one shall be equal to the third angle of the other.

For, adding equals to equals, the two angles of the one triangle together are equal to the two angles of the other. Again, by the propn the three angles of each triangle are together equal to two right angles; and things that are equal to the same thing are equal to one another: therefore the three angles of the one triangle together are equal to the three angles of the other. Hence taking away equals from equals, the remaining angle of the one triangle is equal to the remaining angle of the other. Which was to be proved.

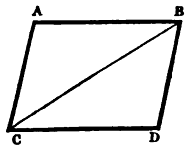
PROP. XXXIII. THEOR.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts shall be also themselves equal and parallel.

Let AB, CD be two equal and parallel straight lines, and

let their extremities be joined towards the same parts, viz. towards A, C and B, D by the straight lines AC, BD. Then AC, BD shall be equal and parallel.

Join BC.



Because BC cuts AB, CD, which are parallels by hyp[s], in B and C, the alternate angles ABC, BCD are equal (i. 29). Also AB is equal to CD by hyp[s], and BC common to the two triangles ABC, BCD; therefore these two triangles have the two sides AB, BC respectively equal to the two sides DC, CB, and the included angle ABC equal to the included angle DCB. Therefore they are equal in every respect (i. 4); and hence the base AC is equal to the base BD, and the angle ACB to the angle DBC. But because BC cutting AC, BD in C, B makes the alternate angles ACB, CBD equal, therefore AC is parallel (i. 27) to BD, and it was above shewn to be equal to BD. Therefore the straight lines AC, BD are equal and parallel. Which was to be proved.

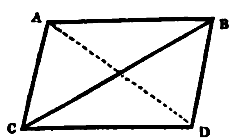
PROP. XXXIV. THEOR.

The opposite sides and angles of parallelograms shall be equal to one another; and each of their diagonals shall bisect them, that is, divide them into two equal triangles.

Let ABDC be a parallelogram. Then :—

I. The opposite sides and angles shall be equal to one another; that is, the side AB shall be equal to the side CD, the side AC to the side BD, the angle BAC to the angle BDC, and the angle ACD to the angle ABD.

Draw one of the diagonals BC.



By the def[n] (Def. A) of a parallelogram AB is parallel to CD, and AC to BD. Now because BC cuts the parallels AB, CD in B, C, the alternate angles ABC, BCD are equal (i. 29); and because BC cuts the parallels AC, BD in C, B, the alternate angles ACB, DBC are equal. Also BC is

common to the two triangles ABC, DCB; therefore these two triangles have the two angles ABC, ACB respectively equal to the two angles DCB, DBC and the sides BC, CB, adjacent to equal angles, equal in each. Therefore they are equal in every respect (I. 26); and hence the side AB is equal to the side CD, the side AC to the side DB, and the angle BAC to the angle CDB. Also it has been proved that the angle ACB is equal to the angle DBC, and that the angle DCB is equal to the angle ABC: therefore adding equals to equals, the whole angle ACD is equal (Ax. 2) to the whole angle DBC. Thus AB has been shewn to be equal to CD, AC to DB, the angle BAC to the angle BDC, and the angle ACD to the angle ABD. Which was to be proved.

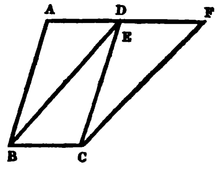
II. Each of the diagonals of the parallelogram ABDC shall bisect it, that is, divide it into two equal triangles.

It was shewn in Part I. of the prop[n] that the triangles ABC, CDB are equal in every respect; and hence the triangle ABC is equal to the triangle CDB, that is, the diagonal BC divides the parallelogram into two equal triangles ABC, BDC. In like manner if the opposite angular points A, D be joined, it may be shewn that the diagonal AD divides the parallelogram into two equal triangles CAD, BDA. Hence each of the diagonals BC, AD of the parallelogram ABCD divides it into two equal triangles, that is, bisects it. Which was to be proved.

PROP. XXXV. THEOR.

Parallelograms on the same base and between the same parallels shall be equal to one another.

Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC. Then these two parallelograms shall be equal to one another.



There are two cases according as the sides AD, EF of the parallelograms, opposite to their common base BC, are terminated in the same point or are not.

I. Let the sides AD, EF be terminated in the same point; the ex-

tremity D of AD coinciding with the extremity E of EF.

Since BD is one of the diagonals of the parallelogram ABCD, it bisects (i. 34) it: therefore the parallelogram ABCD is double of the triangle BDC. And since EC is one of the diagonals of the parallelogram EBCF, it bisects it; therefore the parallelogram EBCF is double of the triangle EBC. But the triangles DBC, EBC are one and the same triangle; and things that are double of the same thing are equal to one another (Ax. 6): therefore the parallelogram ABCD is equal to the parallelogram EBCF.

II. Let the sides AD, EF be not terminated in the same point, either both extremities A, D, of AD falling without EF (Fig. 1), or one of them, as D, falling within EF (Fig. 2).

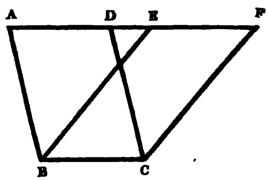


Fig. 1.

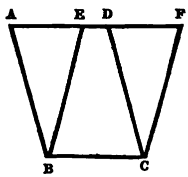


Fig. 2.

Because the opposite sides of parallelograms are equal (i. 34), the side AD of the parallelogram ABCD is equal to the side BC; for the same reason, the side EF of the parallelogram EBCF is equal to the side BC; and things that are equal to the same thing are equal to one another (Ax. 1): therefore AD is equal to EF. To each of these equals AD, EF, in Fig. 1, add DE, then the whole AE is equal (Ax. 2) to the whole DF; and from each of these equals AD, EF, in Fig. 2, take away the common part ED, then the remainder AE is equal (Ax. 3) to the remainder DF. Hence in both figures AE is equal to DF: also, since the opposite sides of parallelograms are equal to one another, the side AB of the parallelogram ABCD is equal to the side CD; and because AB, CD are parallel by the def[n] of a parallelogram, and FA cuts them in A, D, the exterior angle FDC is equal (i. 29) to the interior and opposite angle on the same side, DAB. Therefore the two triangles

EAB, FDC have the two sides EA, AB respectively equal to the two sides FD, DC, and the included angle EAB equal to the included angle FDC. Therefore these two triangles are equal in every respect (i. 4); and hence the triangle EAB is equal to the triangle FDC. From the same magnitude, viz. the figure ABCD, take first the triangle EAB, then the remainder is the parallelogram EBCF; and next take the triangle FDC, then the remainder is the parallelogram ABCD. But these two triangles are equal, and if equals be taken from the same thing the remainders are equal (Ax. 3): therefore the parallelogram EBCF is equal to the parallelogram ABCD.

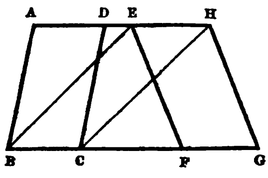
Hence in every case the parallelograms ABCD, EBCF are equal. Which was to be proved.

PROP. XXXVI. THEOR.

Parallelograms on equal bases and between the same parallels shall be equal to one another.

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG. Then the parallelogram ABCD shall be equal to EFGH.

Join BE, CH.



Because the opposite sides of parallelograms are equal (i. 34), EH is equal to FG; and BC is equal to FG by hyp^s; but things that are equal to the same thing are equal to one another (Ax. 1): therefore EH is equal to BC. But it is parallel to BC by hyp^s: therefore EH and BC are equal and parallel straight lines, and they are joined towards the same parts, viz. towards E, B and C, H by the straight lines EB, HC: therefore EB, CH are both equal and parallel (i. 33). And BC is parallel to EH by hyp^s: therefore EBCH is a parallelogram. And because the parallelograms ABCD, EBCH are on the same base BC and between the same parallels BC, AH, ABCD is equal (i. 35) to EBCH; for the same reason, because the parallelograms EFGH, EBCH are on the same base EH and be-

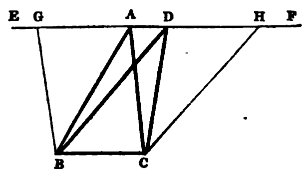
tween the same parallels EH, BG, EFGH is equal to EBCH; and things that are equal to the same thing are equal to one another: therefore the parallelogram ABCD is equal to the parallelogram EFGH. Which was to be proved.

PROP. XXXVII. THEOR.

Triangles on the same base, and between the same parallels shall be equal to one another.

Let the triangles ABC, DBC be on the same base BC and between the same parallels AD, BC. Then the triangle ABC shall be equal to the triangle DBC.

Produce AD both ways to E, F, through B draw (i. 31) BG parallel to CA, cutting AE in G; and through C draw CH parallel to BD, cutting DF in H.

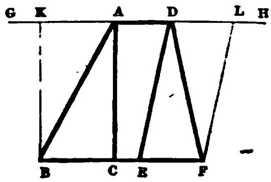


By const^n each of the figures GBCA, DBCH is a parallelogram; and because GBCA, DBGH are parallelograms on the same base BC, and between the same parallels BC, GH, therefore GBCA is equal (i. 35) to DBCH. Again, since AB is one of the diagonals of the parallelogram GBCA, it bisects (i. 34) it; therefore the triangle ABC is half of the parallelogram GBCA: and since DC is one of the diagonals of the parallelogram DBCH, it bisects it, and therefore the triangle DBC is half of the parallelogram DBCH. But the parallelograms GBCA, DBCH were shewn to be equal; and the halves of equal things are equal (Ax. 7): therefore the triangle ABC is equal to the triangle DBC. Which was to be proved.

PROP. XXXVIII. THEOR.

Triangles on equal bases, and between the same parallels, shall be equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD. Then the triangle ABC shall be equal to the triangle DEF.



Produce AD both ways to G and H; through B draw (i. 31) BK parallel to CA, cutting AG in K; and through F draw FL parallel to ED, cutting DH in L.

By constⁿ each of the figures KBCA, DEFL is a parellelogram; and because KBCA, DEFL are parallelograms on equal bases BC, EF and between the same parallels KL, BF, therefore KBCA is equal (i. 36) to DEFL. But since AB is one of the diagonals of the parallelogram KBCA, it bisects (i. 34) it, and the triangle ABC is half of KBCA; and since DF is one of the diagonals of the parallelogram DEFL, it bisects it, and the triangle DEF is half of DEFL. But KBCA, DEFL were shewn to be equal; and the halves of equal things are equal (Ax. 7): therefore the triangle ABC is equal to the triangle DEF. Which was to be proved.

PROP. XXXIX. THEOR.

Equal triangles on the same base, and on the same side of it shall be between the same parallels.

Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it. Then they shall be between the same parallels; that is, if AD be joined, AD shall be parallel to BC.

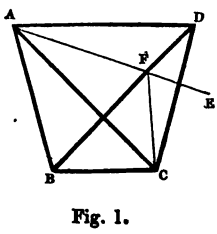


Fig. 1.

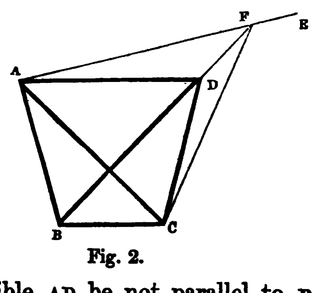


Fig. 2.

For if not; let, if possible, AD be not parallel to BC. Then some other straight line than AD, drawn through A,

will be parallel to BC; let AE be this straight line, and let it cut BD (Fig. 1), or BD produced, if necessary (Fig. 2), in F. Join CF.

Because ABC, FBC are triangles on the same base BC and between the same parallels AF, BC, the triangle ABC is equal (I. 37) to the triangle FBC. But the triangle ABC is equal to the triangle DBC by hyp^e; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the triangle BFC is equal to the triangle BDC; that is, the part equal to the whole (Fig. 1), or the whole equal to the part (Fig. 2): which is impossible (Ax. 9). Therefore AE is not parallel to BC; and it may be shewn in like manner that no other straight line through A but AD is parallel to BC. Hence AD is parallel to BC. Which was to be proved.

PROP. XL. THEOR.

Equal triangles on equal bases in the same straight line, and on the same side of this straight line, shall be between the same parallels.

Let the triangles ABC, DEF be on equal bases BC, EF in the same straight line BF, and on the same side of BF. Then they shall be between the same parallels; that is, if AD be joined, AD shall be parallel to BF.

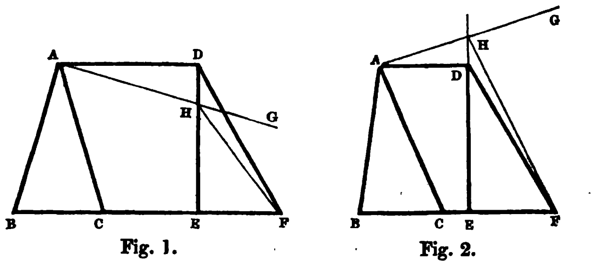


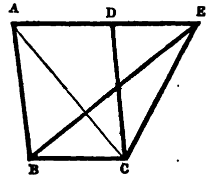
Fig. 1. Fig. 2.

For if not; let, if possible, AD be not parallel to BF. Then some other straight line than AD, drawn through A, will be parallel to BF: let AG be this straight line, and let it cut ED (Fig. 1) or ED produced, if necessary (Fig. 2), in H. Join FH.

Because ABC, HEF are triangles on equal bases BC, EF and between the same parallels AD, BF, the triangle ABC is equal (i. 38) to the triangle HEF. But the triangle ABC is equal to the triangle DEF by hyp^{s}; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the triangle HEF is equal to the triangle DEF; that is, the part equal to the whole (Fig. 1), or the whole equal to the part (Fig. 2): which is impossible (Ax. 9). Therefore AG is not parallel to BF; and it may be shewn in like manner that no other straight line through A but AD is parallel to BF. Hence AD is parallel to BF. Which was to be proved.

PROP. XLI. THEOR.

If a parallelogram and a triangle be on the same base and between the same parallels: then the parallelogram shall be double of the triangle.



Let the parallelogram ABCD and the triangle EBC be on the same base BC, and between the same parallels BC, AE. Then the parallelogram ABCD shall be double of the triangle EBC.

Draw one of the diagonals AC of the parallelogram ABCD.

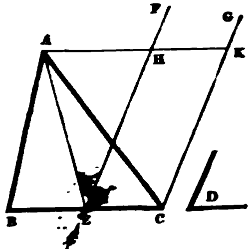
Because AC is a diagonal of the parallelogram ABCD it bisects (i. 34) it, and therefore the parallelogram ABCD is double of the triangle ABC. But because the triangles ABC, EBC are on the same base BC and between the same parallels AE, BC, the triangle ABC is equal (i. 37) to the triangle EBC: therefore the parallelogram ABCD is double of the triangle EBC. Which was to be proved.

PROP. XLII. PROB.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.

Let ABC be the given triangle, and D the given angle.

It is required to describe a parallelogram that shall be equal to the triangle ABC, and have one of its angles equal to D.



Bisect (i. 10) one of the sides of the triangle as BC in E; join AE, and at the point E in the straight line EC make (i. 23) the angle CEF equal to the angle D. Through C draw (i. 31) CG parallel to EF; and through A draw AHK parallel to BC, cutting EF in H, and CG in K. Then the figure HECK, which is a parallelogram by constⁿ, shall be the parallelogram required.

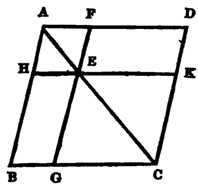
Because the triangles ABE, AEC are on equal bases BE, EC and between the same parallels BC, AK they are equal (i. 38) to one another, and the triangle ABC is double of the triangle AEC. But because the parallelogram HECK and the triangle AEC are on the same base EC and between the same parallels EC, AK, the parallelogram HECK is likewise double (i. 41) of the triangle AEC; and things that are double of the same thing are equal to one another (Ax. 6): therefore the parallelogram HECK is equal to the triangle ABC, and one of its angles, HEC, is equal to the angle D by constⁿ. Hence there has been described a parallelogram HECK equal to the given triangle ABC, and having an angle HEC equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

If through any point in one of the diagonals of a parallelogram, two straight lines be drawn parallel to its sides, so as to divide it into four parallelograms, two of which are about the diagonal of the parallelogram (i. e. have the parts of the diagonal for their diagonals), and the other two are the complements of these (i. e. make up together with them the whole figure); then the complements shall be equal.

Let AC be one of the diagonals of the parallelograms

ABCD, and E any point in AC; through E let FEG be drawn parallel to AB or CD, cutting AD in F and BC in G, and HEK parallel to AD or BC, cutting AB in H and CD in K, so as to divide ABCD into four parallelograms, of which the two AHEF, EGCK are about the diagonal AC (i. e. have the parts AE, EC, for their respective diagonals), and the other two HEGB, FDKE are their complements (i. e. together with AHEF, EGCK make up the whole figure ABCD). Then the complement HBGE shall be equal to the complement FEKD.



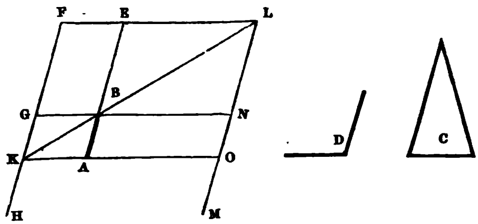
Because AC is a diagonal of the parallelogram ABCD, it bisects (i. 34) it, and the triangle ABC is equal to the triangle ADC. Again, because AE is a diagonal of the parallelogram AHEF it bisects it, and the triangle AHE is equal to the triangle AFE; and because EC is a diagonal of the triangle EGCK it bisects it, and the triangle EGC is equal to the triangle EKC: therefore, adding equals to equals, the triangle AHE, together with the triangle EGC is equal (Ax. 2) to the triangle AFE together with the triangle EKC. But it was shewn that the whole triangle ABC is equal to the whole triangle ADC: therefore, taking away equals from equals, the remaining complement HBGE is equal (Ax. 3) to the remaining complement FEKD. Which was to be proved.

PROP. XLIV. PROB.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle and have one of its angles equal to a given angle.

Let AB be the given straight line, C the given triangle, and D the given angle. It is required to apply to the straight line AB a parallelogram (i. e. to describe a parallelogram having AB for one of its sides), which shall be equal to the triangle C and have one of its angles equal to the angle D.

Describe (i. 42) the parallelogram EFGB equal to the triangle C, and having one of its angles EBG equal to the



angle D, in such a situation that the side EB shall be contiguous to and in the same straight line with AB. Produce FG to H; through A draw (i. 31) AK parallel to BG or EF, cutting FH in K; and join KB. Then because FH cuts the parallels FE, KA in F, K, the two interior angles on the same side of FH, EFK, FKA are equal (i. 29) to two right angles; and therefore the angles EFK, FKB are less (Ax. 9) than two right angles. Hence the straight line FH cutting the two straight lines FE, KB in F, K makes the two interior angles EFK, FKB on the same side of FH together less than two right angles: therefore by the axiom (Ax. 12) FE, KB, being continually produced, shall at length meet in some point on the side of FH towards E, B. Let them be produced to meet in L. Through L draw LM parallel to EA, or FK; and produce GB, KA to cut LM in N, O. Then the figure ABNO, which by constⁿ is a parallelogram, shall be the parallelogram required.

Because FKOL is a parallelogram by constⁿ, and EA, GN are drawn through the point B in one of its diagonals KL, so as to divide it into the two parallelograms about its diagonal, GKAB, EBNL, and the complements FLBG, BAON: therefore the complement FB is equal (i. 43) to the complement BO. But FB is equal to the triangle C by constⁿ; and things that are equal to the same thing are equal to one another (Ax. 1); therefore BO is equal to the triangle C. Also the angle ABN is equal to the angle GBE, since they are opposite vertical angles (i. 15), and the angle D is equal to the angle GBE by constⁿ; and things that are equal to the same thing are equal to one another: therefore the angle ABN is equal to the angle D, and BO has AB for one of its sides.

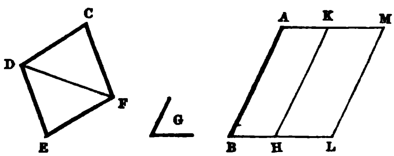
Hence to the given straight line AB a parallelogram ABNO has been applied, equal to the given triangle C and having one of its angles ABN equal to the given angle D. Which was to be done.

PROP. XLV. PROB.

To a given straight line to apply a parallelogram which shall be equal to a given polygon, and have an angle equal to a given angle.

I. Let the polygon have four sides.

Let AB be the given straight line, CDEF the given polygon of four sides, G the given angle. It is required to apply to AB a parallelogram, which shall be equal to the polygon CDEF, and have one of its angles equal to G.



Join two of the opposite angular points as DF of the polygon, so as to divide it into the two triangles CDF, DFE. To the straight line AB apply (i. 44) the parallelogram ABHK, equal to the triangle CDF, and having one of its angles ABH equal to the angle G; and to the straight line KH apply the parallelogram KHLM equal to the triangle DEF, and having the angle KHL equal to the angle G. Then the whole figure AL shall be a parallelogram, and such as is required.

Because by const^{n} each of the angles ABH, KHL is equal to the angle G; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle ABH is equal to the angle KHL. To each of these equals add the angle BHK; therefore the angles ABH, BHK are equal (Ax. 2) to the angles KHL, BHK. But because BH cuts the parallels AB, KH in B, H, the two interior angles ABH, BHK on the same side of BH are equal (i. 29) to two right angles; and things that are equal to the

same thing are equal to one another: therefore also the angles KHL, KHB are equal to two right angles. That is, at the point H in the straight line HK, the two straight lines BH, LH make with HK the adjacent angles on the opposite sides of HG, KHL, KHB equal to two right angles; therefore BH is in the same straight line (I. 14) with HL:

Again, because HK cuts the parallels BL, AK in H, K, the alternate angles KHL, HKA are equal; to each of these equals add the angle HKM; then the angles KHL, HKM are equal to the angles HKA, HKM. But because HK cuts the parallels KM, HL in K, H, the interior angles HKM, KHL on the same side of HK are equal to two right angles; and things that are equal to the same thing are equal to one another; therefore the angles HKA, HKM are equal to two right angles. That is, at the point K in the straight line KH, the two straight lines KA, KM, on opposite sides of KH, make with KH the adjacent angles AKH, MKH equal to two right angles; therefore, KA is in the same straight line with KM:

But it was shewn that BH, HL were in the same straight line, and BH is parallel to AK: therefore the two straight lines AKM, BHL are parallel. Also AB is parallel to KH, and ML is parallel to KH; and two straight lines, which are each of them parallel to the same straight line, are parallel to one another (I. 30): therefore AB is parallel to ML. Hence the figure AMLB is a parallelogram; and AB is one of its sides, and by const^{n} one of its angles ABL is equal to the angle G:

Lastly, the parallelogram AH is equal to the triangle CDF by const^{n}, and the parallelogram KL is equal to the triangle DEF for the same reason: therefore adding equals to equals, the parallelogram AL is equal to the whole polygon CDEF.

Hence to the straight line AB has been applied a parallelogram ABLM equal to the given polygon CDEF, and having one of its angles ABL equal to the given angle G. Which was to be done.

II. Let the polygon have more than four sides.

The polygon must be divided into triangles by joining

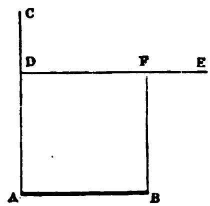
one of its angular points with all the others; and successive parallelograms be applied to one another, equal to the different triangles of the polygon, just as KL was to AH in Case I. And it can be shewn by a similar proof that the resulting figure is a parallelogram fulfilling the requisites of the problem.

PROP. XLVI. PROB.

To describe a square on a given straight line.

Let AB be a given straight line. It is required to describe a square on AB.

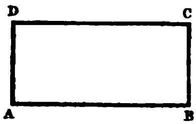
From A draw (i. 11) AC at right angles to AB; and from AC cut off (i. 3) AD equal to AB. Through D draw (i. 31) DE parallel to AB; and through B draw BF parallel to AC, cutting DE in F. Then ABFD shall be the square required.



The figure ABFD is a parallelogram by const^{n}; and the opposite sides of parallelograms are equal (i. 34): therefore AB is equal to DF, and AD to BF. But AB is equal to AD by const^{n}: therefore the four straight lines DA, AB, BF, FD are all equal, and the figure ABFD is equilateral. Again, because AC cuts the parallels DF, AB in D, A, the two interior angles on the same side of DA, ADF, DAB are equal (i. 29) to two right angles. But DAB is a right angle by const^{n}: therefore also ADF is a right angle; and the opposite angles of parallelograms are equal (i. 34); therefore each of the opposite angles ABF, BFD is a right angle: wherefore the figure ABFD is rectangular. Hence the four-sided figure ABFD has been shewn to be both equilateral and rectangular, and it is therefore a square (Def. 30), and it is described on the given straight line AB. Which was to be done.

COR.—Every parallelogram that has one of its angles a right angle shall have all its angles right angles.

Let ABCD be a parallelogram, and let it have one of its angles DAB a right angle. Then the three other angles ABC, BCD, CDA shall also be right angles.



Because AD cuts the parallels AB, DC in AD, it may be shewn as in the propn that the angle ADC is also a right angle, and then that each of the other two angles opposite is a right angle. Hence ABCD is rectangular. Which was to be proved.

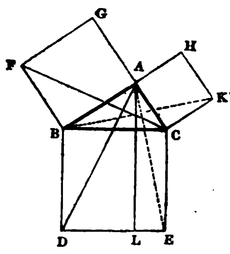
OBS.—It appears from the proof of this propn that the opposite sides of a square are parallels.

PROP. XLVII. THEOR.

In any right-angled triangle the square, which is described on the side opposite to the right angle, shall be equal to the squares, which are described on the sides including the right angle.

Let ABC be a right-angled triangle having the right angle BAC. Then the square described on the side BC, opposite to the right angle BAC, shall be equal to the squares described on the sides AB, AC, including the right angle.

On BC, AB, AC describe (i. 46) the three squares BDEC, BFGA, AHKC; through A draw (i. 31) AL parallel to BD or CE (i. 46, Obs.), cutting DE in L; and join FC, AD.



Because BAC is a right angle by hyps, and BAG is a right angle, since it is an angle of a square (Def. 30); therefore the two straight lines CA, GA on opposite sides of BA make with it at the point A the adjacent angles BAC, BAG equal to two right angles. Therefore CA is in the same straight line with AG. In like manner it may be shewn that BA is in the same straight line with AH. Because DBC is a right angle, since it is an angle of a square, and

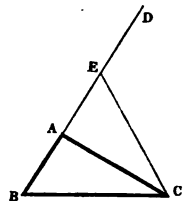
ABF is a right angle for the same reason; and all right angles are equal to one another (Ax. 11): therefore the angle DBC is equal to the angle FBA. To each of these angles add the angle ABC; then the whole angle DBA is equal (Ax. 2) to the whole angle FBC. Also AB is equal to FB, and BD to BC, since they are sides of squares (Def. 30): therefore the two triangles DBA, FBC have the two sides AB, BD respectively equal to the two sides FB, BC and the included angle ABD equal to the included angle FBD. Therefore these two triangles are equal in every respect (i. 4), and hence the triangle ABD is equal to the triangle FBC. Now because the parallelogram (i. 46, Obs.) BL and the triangle ABD are on the same base BD and between the same parallels BD, AL; therefore the parallelogram BL is double (i. 41) of the triangle ABD. And because the square GB and the triangle CFB are on the same base FB and between the same parallels FB, GAC; therefore the square GB is double of the triangle BFC. But the triangle ABD was shewn to be equal to the triangle FBC; and the doubles of equal things are equal (Ax. 7): therefore the parallelogram BL is equal to the square GB. And in like manner by joining AE, BK it may be shewn that the parallelogram CL is equal to the square HC. Therefore, adding equals to equals, the whole square BDEC is equal to the two squares GB, HC; that is, the square described on the side BC, opposite to the right angle, is equal to the squares described on the sides AB, AC, including the right angle. Which was to be proved.

PROP. XLVIII. THEOR.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides: then the angle included by these two sides shall be a right angle.

Let the square described on BC one of the sides of the triangle ABC be equal to the squares described on the other two sides AB, AC. Then the angle BAC included by AB, AC shall be a right angle.

From A draw (I. 11) AD at right angles to AC: from AD cut off (I. 3) AE equal to AB; and join CE.



Because AE is equal to AB by const^n, the square of AE (i. e. the square described on AE) is equal to the square of AB. To each of these equals add the square of AC: therefore the squares of EA, AC are equal (Ax. 2) to the squares of BA, AC. But the square of EC is equal (I. 47) to the squares of EA, AC, because the angle CAE is a right angle by const^n; and the square of BC is equal to the squares of BA, AC by hyp^s: therefore the square of EC is equal to the square of BC, and therefore EC is equal to BC. Also EA is equal to AB, and AC common to the two triangles EAC, BAC: therefore these two triangles have the three sides CE, EA, AC respectively equal to the three sides CB, BA, AC. Therefore they are equal in every respect (I. 8); and hence the angle CAE is equal to the angle CAB. But CAE is a right angle: therefore also BAC is a right angle. Which was to be proved.

THE

ELEMENTS OF EUCLID.

BOOK II.

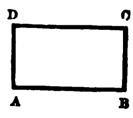
DEFINITIONS.

I.

A RECTANGLE is a parallelogram which has one of its angles a right angle (and therefore (i. 46, Cor.) all its angles right angles).

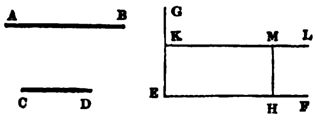
OBS. A rectangle is said to be contained by either pair of its sides which include one of the right angles.

(1) Let ABCD be a rectangle. We may either speak of it as "the rectangle contained by AB, AD, or "the rectangle contained by BA, BC," or "the rectangle contained by CB, CD," or "the rectangle contained by DA, DC."



Frequently the words "contained by" are not expressed, but understood, so that the above rectangle may be simply called either "the rectangle AB, AD," or "the rectangle BA, BC," or "the rectangle CB, CD," or "the rectangle DA, DC."

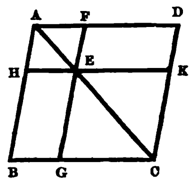
(2) When the rectangle contained by two straight lines is spoken of, there is not always a figure given. Thus, if the "rectangle contained by the straight lines AB, CD," or simply "the rectangle AB, CD," be spoken of, and we wish to describe it; we must take some straight line EF, and



draw (i. 11) from E another, EG, at right angles to it. From EF cut off (i. 3) EH equal to AB, and from EG cut off EK equal to CD; through K draw (i. 31) KL parallel to EF, and through H, HM parallel to EG, cutting KL in M. Then the figure KEHM is a rectangle by const^n, and it is the rectangle contained by EH, EK, that is, by AB, CD.

II.

If through any point in one of the diagonals of a parallelogram two straight lines be drawn parallel to its sides, so as to divide it into two parallelograms about the diagonal and two complements, either of the two parallelograms about the diagonal together with both the two complements make up a figure which is called a gnomon.



OBS. Let ABCD be a parallelogram, AC one of its diagonals, E any point in AC; through E let FEG be drawn parallel to AB or DC, cutting AD in F, and BC in G, and HEK parallel to AD or BC, cutting AB in H and DC in K; so as to divide ABCD into the parallelograms HF, GK about its diagonal AC, and the two complements HG, FK.

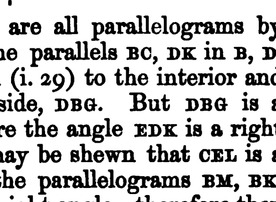
Then according to the def^n, the parallelogram HF together with HG and FK, or the figure DABGEK is one gnomon, which is called the gnomon KFB or the gnomon DHG: and the other parallelogram GK together with HG and FK, or the figure DFEHBC is the other gnomon, which is called the gnomon FKB, or the gnomon DGH.

PROPOSITIONS.

PROP. I. THEOR.

If there be two straight lines, and one of them be divided into any number of parts: then the rectangle contained by the two straight lines shall be equal to the rectangles contained by the undivided and the several parts of the divided line.

Let A and BC be two straight lines, and let BC be divided into any number of parts BD, DE, EC, in D, E. Then the rectangle contained by A, BC shall be equal to the rectangle contained by A, BD together with that contained by A, DE and that contained by A, EC.



From B draw (i. 11) BF at right angles to BC, and from BF cut off (i. 3) BG equal to A. Through G draw (i. 31) GH parallel to BC, and through D, E, C draw DK, EL, CM parallel to BF, cutting GM in K, L, M.

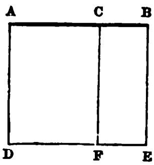
The figures BM, BK, DL, EM are all parallelograms by constⁿ; and because BC cuts the parallels BC, DK in B, D, the exterior angle CDK is equal (i. 29) to the interior and opposite angle on the same side, DBG. But DBG is a right angle by constⁿ; therefore the angle EDK is a right angle, and in like manner it may be shewn that CEL is a right angle. Hence each of the parallelograms BM, BK, DL, EM has one of its angles a right angle; therefore they are all rectangles (i. 46, Cor.). Now the rectangle BM is equal to the rectangles BK, DL, EM. But BM is contained by A, BC, for it is contained by GB, BC, and GB is equal to

A by constⁿ; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, for it is contained by DK, DE, and DK is equal to A, since DK is equal to BG, and BG is equal to A; and in like manner EM is contained by A, EC. Therefore the rectangle contained by A, BC is equal to the several rectangles contained by A, BD, and by A, DE, and by A, EC. Which was to be proved.

PROP. II. THEOR.

If a straight line be divided into any two parts: then the rectangles contained by the whole line and each of the parts shall be together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in C. Then the rectangle contained by AB, BC together with that contained by AB, AC shall be equal to the square of AB.



On AB describe (i. 46) the square ADEB; and through C draw (i. 31) CF parallel to AD or BE, cutting DE in F.

The figures AF, CE are parallelograms by $const^n$; and since each of them contains an angle of a square, which is a right angle (Def. 30), they are rectangles (i. 46, Cor.). Now, AE is equal to AF and CE. But AE is the square of AB; AF is the rectangle AB, AC; for it is the rectangle contained by AD, AC, and AC is equal to AB, since they are sides of a square; and CE is the rectangle AB, BC, for it is contained by CB, BE, and BE is equal to BA, since they are sides of a square. Therefore the rectangle AB, AC together with the rectangle AB, BC is equal to the square of AB. Which was to be proved.

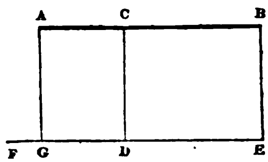
PROP. III. THEOR.

If a straight line be divided into any two parts: then the rectangle contained by the whole line and one of the parts, shall be equal to the rectangle contained by the

two parts together with the square of the aforesaid part.

Let the straight line AB be divided into any two parts in C. Then the rectangle AB, BC shall be equal to the rectangle AC, CB together with the square of BC: and the rectangle AB, AC shall be equal to the rectangle AC, CB together with the square of AC.

I. The rectangle AB, BC shall be equal to the rectangle AC, CB together with the square of BC.



On BC describe (i. 46) the square BCDE; produce ED to F; and through A draw (i. 31) AG parallel to BE or CD, cutting EF in G.

The figures AE, AD are parallelograms by const^{n}; and since they have each the angles ABE, ACD, which are right angles, they are also rectangles (i. 46, Cor.). Now AE is equal to AD and CE. But AE is the rectangle AB, BC, for it is contained by AB, BE, and BE is equal to BC, since they are sides of a square (Def. 30); also AD is the rectangle AC, CB, for it is contained by AC, CD, and CD is equal to CB, since they are sides of a square; and CE is the square of BC. Therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the square of BC. Which was to be proved.

II. The rectangle AB, AC shall be equal to the rectangle AC, CB together with the square of AC.

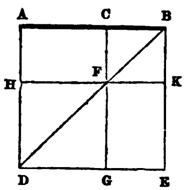
This may be shewn by a construction and demonstration exactly similar to those in (I.), the const^{n} here commencing by describing the square on AC, instead of BC. Which was to be proved.

PROP. IV. THEOR.

If a straight line be divided into any two parts: then the square of the whole line shall be equal to the squares of the two parts together with twice the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts

AC, CB in C. Then the square of AB shall be equal to the squares of AC, CB together with twice the rectangle AC, CB.



On AB describe (i. 46) the square ADEB; and draw one of its diagonals BD. Through C draw (i. 31) CFG parallel to AD or BE, cutting BD in F, and DE in G; and through F draw HFK parallel to AB or DE, cutting AD in H, and BE in K.

The square ADEB is divided by the constⁿ into four parallelograms, two of which HG, CK are parallelograms about its diagonal BD, and the other two AF, FE are their complements. Hence the complement AF is equal (i. 43) to the complement FE; and since each of these four parallelograms has one angle of the square ADEB, and therefore a right angle (Def. 30) for one of its angles, they are all rectangular (i. 46, Cor.). Again because BD cuts the parallels AD, CG in D, F, the exterior angle CFB is equal (i. 29) to the interior and opposite angle ADF on the same side of BD; but the angle ABD is likewise equal (i. 5) to the angle ADF, because AB is equal to AD, since they are sides of a square (Def. 30); and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle CFB is equal to the angle CBF, and therefore CB is equal (i. 6) to CF. But the opposite sides of parallelograms are equal to one another (i. 34); therefore CF is equal to FK, and CF to BK: hence the four sides of the parallelogram CFKB are equal to one another, and it was shewn to be rectangular. Therefore it is a square and it is described on BC. Similarly it may be shewn that HG is a square, and it is described on HF; but HF is equal to AC since the opposite sides of parallelograms are equal: hence HG is equal to the square of AC. And AF is the rectangle AC, CB, for it is contained by AC, CF of which CF has been proved equal to CB: therefore also FE, which is equal to AF, is equal to the rectangle AC, CB; and therefore AF, FE together are equal to twice the rectangle AC, CB. Hence HG, CK, AF, FE are equal to the squares of AC, CB and twice the rectangle AC, CB; and HG, CK, AF, FE make up

the whole figure ADEB, which is the square of AB. Therefore the square of AB is equal to the squares of AC, CB together with twice the rectangle AC, CB. Which was to be proved.

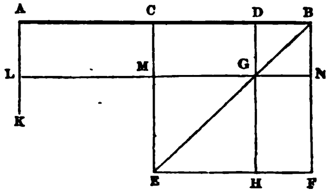
COR.—If through any point in one of the diagonals of a square, two straight lines be drawn parallel to the sides, so as to divide it into four parallelograms two of which are about its diagonal, and two are complements: then the two parallelograms about the diagonal of the square shall be squares, and the two complements shall be rectangles.

For let ADEB be a square; BD one of its diagonals: through F any point in BD, let CFG be drawn parallel to AD or BE, cutting AB in C, DE in G; and let HFK be drawn parallel to AB or DE, cutting AD in H, BE in K. Then it may be shewn exactly as in the prop^{n} that HG, CK, the parallelograms about the diagonal BD of the square, are squares, and that their two complements are rectangles. Which was to be proved.

PROP. V. THEOR.

If a straight line be bisected, and also divided into two unequal parts: then the rectangle contained by the unequal parts together with the square of the part of the line intercepted between the points of section, shall be equal to the square of half the line.

Let the straight line AB be bisected in C, and divided into two unequal parts in D. Then the rectangle AD, DB together with the square of CD shall be equal to the square of BC.



On CB describe (i. 46) the square CEFB, and draw the diagonal BE. Through D draw (i. 31) DGH parallel to CE or BF, cutting BE in G and EF in H; through A draw AK parallel to CE or BF; and

through G draw LMGN parallel to CB or EF, cutting AK in L, CE in M, BF in N.

The square CF is divided by the const^{n} into four parallelograms, two of which MH, DN are about its diagonal BE, and the two CG, GF are their complements. Hence CG, GF are equal (i. 43); they are both rectangles (ii. 4, Cor.) and MH, DN are both squares. Again, because DA cuts the parallels AL, CM in A, C, the exterior angle DCM is equal (i. 29) to the interior and opposite angle on the same side, CAL; but DCM is an angle of a square and therefore a right angle (Def. 30): hence also CAL is a right angle, and therefore ALGD, which is a parallelogram by const^{n}, is a rectangle (i. 46, Cor.). Now, since CG is equal to GF, to each of these equals add the square DN; then the whole CN is equal (Ax. 2) to the whole DF. But because the parallelograms CN, AM are on equal bases CB, AC and between the same parallels AB, LN, therefore CN is equal (i. 36) to AM; and things that are equal to the same thing are equal to one another (Ax. 1): therefore AM is equal to DF. To each of these equals add CG; then the whole AG is equal to the gnomon CNH. But AG is the rectangle AD, DB, for it is contained by AD, DG, and DG is equal to DB, since they are sides of a square (Def. 30): therefore the gnomon CNH is equal to the rectangle AD, DB. To each of these equals add MH, which is the square of CD, for it is the square of MG, and CD is equal to MG, since the opposite sides of parallelograms are equal (i. 34): then the gnomon CNH together with MH is equal to the rectangle AD, DB together with the square of CD. But the gnomon CNH together with MH makes up the whole figure CEFD, which is the square of CB. Hence the rectangle AD, DB together with the square of CD is equal to the square of BC. Which was to be proved.

COR.—If there be two unequal straight lines: then the rectangle contained by the straight line which is equal to their sum, and the straight line which is equal to the excess of the greater above the less shall be equal to the excess of the square of the greater above the square of the less.

With the figure of the propⁿ, let AC be the greater, and CD the less of two unequal straight lines, which are placed contiguous and in the same straight line; and in this straight line let CB be equal to AC: so that AD is the sum of AC and CD, and DB which together with CD makes up BC is the excess of BC, that is, of AC above CD. Then shall the rectangle AD, DB be equal to the excess of the square of AC above the square of CD.

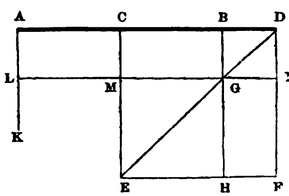
Because AB is bisected in C, and divided into two unequal parts in D, therefore by the propⁿ, the rectangle AD, DB together with the square of CD is equal to the square of BC. Hence the square of BC, or of AC, is greater than the square of CD by the rectangle AD, DB; that is, the rectangle AD, DB is equal to the excess of the square of AC above the square of CD. Which was to be proved.

PROP. VI. THEOR.

If a straight line be bisected, and produced to any point: then the rectangle contained by the whole line thus produced and the part of it produced, together with the square of half the bisected line, shall be equal to the square of the line, which is made up of the half and the part produced.

Let the straight line AB be bisected in C and produced to D. Then the rectangle AD, DB together with the square of CB shall be equal to the square of CD.

On CD describe (I. 46) the square CEFD, and draw the diagonal DE. Through B draw (I. 31) BGH parallel to CE or DF, cutting DE in G, and EF in H; through A draw AK parallel to CE or DF; and through G draw LMGN parallel to CD or EF, cutting AK in L, CE in M, and DF in N.



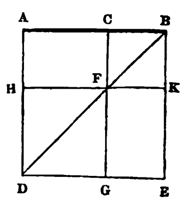
Then it may be shewn as in the preceding propⁿ that CG, GF are equal; that they are both rectangles, and MH, BN both squares; and that AN is a rectangle. Now because the parallelograms AM, CG are on equal bases AC,

CB and between the same parallels AB, LG, therefore AM is equal (I. 36) to CG; but GF is equal to CG; and things that are equal to the same thing are equal to one another (Ax. 1): therefore AM is equal to GF. To each of these equals add CN: therefore the whole AN is equal (Ax. 2) to the gnomon CNH. But AN is the rectangle AD, DB; for it is contained by AD, DN, and DB is equal to DN, since they are sides of the square BN (Def. 30): therefore the gnomon CNH is equal to the rectangle AD, DB. To each of these equals add MH, which is equal to the square of BC, for it is described on MG, and MG is equal to BC, since the opposite sides of parallelograms are equal (I. 34); therefore the gnomon CNH together with MH is equal to the rectangle AD, DB together with the square of BC. But the gnomon CNH and MH make up the whole figure CF, which is the square of CD. Hence the rectangle AD, DB together with the square of BC is equal to the square of CD. Which was to be proved.

PROP. VII. THEOR.

If a straight line be divided into any two parts: then the squares of the whole line and of one of the parts shall be equal to twice the rectangle contained by the whole line and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts AC, CB in C. Then the squares of AB, BC shall be equal to twice the rectangle contained by AB, BC together with the square of AC; and the squares of AB, AC shall be equal to twice the rectangle AB, AC together with the square of BC.



On AB describe (I. 46) the square ADEB, and construct the figure as in Prop. IV.

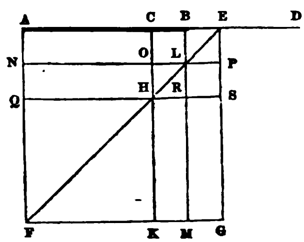
The square ADEB is divided by the constⁿ into four parallelograms, two of which HG, CK are about its diagonal BD, and two AF, FE are their complements. Hence AF is equal to FE (I. 43), and they, as well as AK, CE are

rectangles; and HG, CK are squares (ii. 4, Cor.). Since AF is equal to FE, to each of these equals add CK: then the whole AK is equal (Ax. 2) to the whole CE; and AK, CE are double of AK. But AK, CE are the gnomon AKG together with CK; and AK is the rectangle AB, BC, for it is contained by AB, BK, and BK is equal to BC, since they are sides of the square CK (Def. 30): therefore the gnomon AKG together with CK is equal to twice the rectangle AB, BC. To each of these equals add HG, which is equal to the square of AC, for it is the square described on HF, and HF is equal to AC, because the opposite sides of parallelograms are equal to one another (i. 34); therefore the gnomon AKG together with the squares CK, HG is equal to twice the rectangle AB, BC and the square of AC. But the gnomon AKF together with the squares CK, HG makes up the whole figure AE and CK, which are the squares of AB and BC. Hence the squares of AB and BC are equal to twice the rectangle AB, BC together with the square of AC. And in the same manner it might be shewn that the squares of AB and AC are equal to twice the rectangle AB, AC together with the square of BC. Which was to be proved.

PROP. VIII. THEOR.

If a straight line be divided into any two parts: then four times the rectangle contained by the whole line and one of the parts, together with the square of the other part shall be equal to the square of the line which is made up of the whole line and that part.

Let the straight line AB be divided into any two parts AC, CB in C. Then four times the rectangle AB, BC together with the square of AC shall be equal to the square of the straight line made up of AB and BC together; and four times the rectangle BA, AC together with the square



of BC shall be equal to the square of the straight line which is made up of BA and AC together.

Produce AB to D, and from BD cut off (I. 3) BE equal to BC. On AE describe (I. 46) the square AFGE, and draw the diagonal EF. Through C draw (I. 31) CHK parallel to AF or EG cutting EF in H, and FG in K; through B draw BLM parallel to AF or EG, cutting EF in L and FG in M; through L draw NOLP parallel to AE or FG, cutting AF in N, CK in O, and EG in P; and lastly through H draw QHRS parallel to AE or FG cutting AF in Q, BM in R, and EG in S.

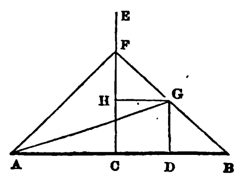
The square AFGE is divided by the constⁿ into four parallelograms, two of which QK, CS are about the diagonal EF and the two AH, HG are their complements: hence AH is equal to HG (I. 43), and they are both rectangles, and QK, CS are both squares (I. 46, Cor.). Again the square CHSE is divided by the constⁿ into four parallelograms, two of which BP, OR are about the diagonal EH, and the two CL, LS are their complements; hence CL is equal to LS, and they are both rectangles, and BP, OR are squares. Now because CB is equal to BE by constⁿ, the rectangles CL, BP are on equal bases CB, BE and between the same parallels, therefore CL is equal (I. 36) to BP; and because the opposite sides of parallelograms are equal (I. 34), OL is equal to CB, and LP is equal to BE, therefore OL is equal to LP, and OR, LS are parallelograms on equal bases OL, LP and between the same parallels. Hence OR is equal to LS; and CL was shewn to be equal to LS: therefore CL, LS, BP, OR are all equal, and the whole CHSE is four times one of them CL. Again, since OR is a square, the side OL is equal (Def. 30) to OH; and since BP is a square, the side LP is equal to BL, and therefore to CO: but it was proved that OL is equal to LP, therefore also CO is equal to OH, and therefore AO, NH are rectangles on equal bases CO, OH and between the same parallels AQ, CH: hence AO is equal to NH, and AQHC is double of AO. But AQHC is equal to HKGS: therefore likewise HKGS is double of AO. Hence AQHC and HKGS are together four times AO; and it has been shewn that CHSE is four times CL: therefore the whole gnomon ASK is four times the whole AL. But AL is the rectangle AB, BC, for it is con-

tained by AB, BL, and BL is equal to BE, or BC; therefore the gnomon ASK is four times the rectangle AB, BC. To each of these equals add QK, which is the square of AC, for it is the square described on QH, and QH is equal to AC, since the opposite sides of parallelograms are equal: then the gnomon ASK together with QK is equal (Ax. 2) to four times the rectangle AB, BC together with the square of AC. But the gnomon ASK together with QR makes up the whole figure AFGE, which is the square of AE, that is, of the line made up of AB and BC, since BE is equal to BC by constⁿ: therefore four times the rectangle AB, BC together with the square of AC is equal to the square of the line made up of AB and BC. And in a similar manner by producing BA and proceeding in the constⁿ as before, it may be shewn that four times the rectangle BA, AC together with the square of BC is equal to the square of the line made up of BA and AC. Which was to be proved.

PROP. IX. THEOR.

If a straight line be bisected, and also divided into two unequal parts: then the squares of the two unequal parts shall be double of the square of half the line, and the square of the line between the points of section.

Let the straight line AB be bisected in C and divided into two unequal parts AD, DB in D. Then the squares of AD, DB shall be double of the squares of AC, CD.



From C draw (I. 11) CE at right angles to AB, and from CE cut off (I. 3) CF equal to AC or CB. Join FA, FB; through D draw (I. 31) DG parallel to CE cutting BF in G; through G draw GH parallel to AB cutting CF in H; and join AG.

Because FC cuts the parallels GH, CB in H, C, the exterior angle FHG is equal (I. 29) to the interior and opposite angle HCD on the same side of FC; but HCD is a right angle by constⁿ: therefore also FHG is a right

angle. Again because BC cuts the parallels GD, HC in D, C, the exterior angle GDB is equal to the interior and opposite angle on the same side of BC, HCB; but HCB is a right angle by constn: therefore also GDB is a right angle. The three angles CAF, AFC, FCA of the triangle ACF are equal (I. 32) to two right angles, and ACF is a right angle by constn, therefore the two angles CAF, AFC are equal to one right angle; but because CA is equal to CF, the angle CAF is equal (I. 5) to the angle CFA; hence each of the angles CAF, CFA is half a right angle. For a like reason each of the angles CBF, CFB is half a right angle: therefore the whole angle AFB is a right angle. Again the three angles HFG, FGH, GHF of the triangle FHG are equal to two right angles; and the angle GHF has been proved to be a right angle; therefore the two angles HFG, FGH are equal to one right angle; but the angle HFG has been shewn to be half a right angle, therefore the remaining angle HGF is also half a right angle, and the angle HFG is equal to the angle HGF: hence HF is equal (I. 6) to HG. Also the three angles DGB, GBD, BDG of the triangle BDG are equal to two right angles; and GDB has been proved to be a right angle; therefore the two angles DGB, GBD are equal to one right angle; but the angle DBF has been proved to be half a right angle, therefore the remaining angle DGB is also half a right angle, and the angle DGB is equal to the angle DBG: hence DG is equal to DB:

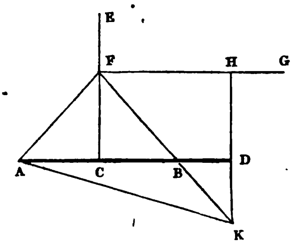
Now because AC is equal to CF, the square of AC is equal to the square of CF; and the square of AF is equal to the squares of AC, CF (I. 49), because ACF is a right angle: therefore the square of AF is double of the square of AC. Again because FHG is a right angle, the square of FG is equal to the squares of FH, HG, and because FH is equal to HG, the square of FH is equal to the square of HG: therefore the square of FG is double of the square of HG: that is, of the square of CD, because HG is equal to CD, since the opposite sides of parallelograms are equal (I. 34). But the square of AF is double of the square AC: therefore the squares of AF, FG are double of the squares of AC, CD; but the square of AG is equal to the squares of AF, FG, because the angle AFG is a right angle:

therefore the square of AG is double of the squares AC, CD. Now since DB is equal to DG, the square of DB is equal to the square of DG; to each of these equals add the square of AD; therefore the squares of AD, DB are equal (Ax. 2) to the squares of AD, DF; but because ADG is a right angle, the square of AG is equal to the squares of AD, DG; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the square of AG is equal to the squares of AD, DB. And it has been shewn that the square of AG is double of the squares of AC, CD; hence also the squares of AD, DB are double of the squares of AC, CD. Which was to be proved.

PROP. X. THEOR.

If a straight line be bisected, and produced to any point: then the squares of the whole line thus produced and the part of it produced shall be double of the squares of half the bisected line, and the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D. Then the squares of AD, DB shall be double of the squares of AC, CD.



From C draw (I. 11) CE at right angles to AB, and from CE cut off (I. 3) CF equal to AC, or CB. Join AF, FB; through F draw (I. 31) FG parallel to AD; and through D draw DH parallel to CE, cutting FG in H. Then because FG cuts the parallels CE, DH in F, H, the two interior angles CFH, FHD on the same side of FG are equal (I. 29) to two right angles; and therefore the angles BFH, FHD are less (Ax. 9) than two right angles. Hence the straight line FH, cutting the straight lines BF, DH in F, H, makes the two interior angles BFH, FHD on the same side of FH together less than two right angles: therefore by the axiom (Ax. 12) FB, HD, being con-

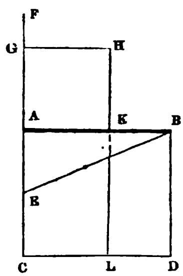
tinually produced, shall at length meet in some point on the side of FH towards B, D. Let them be produced to meet in K; and join AK.

The figure FCDH is by constⁿ a parallelogram, and one of its angles FCD is a right angle; therefore it is rectangular (I. 46, Cor.), and hence the angles FHK, CDH, and its adjacent (Def. 10) angle CDK are right angles. It may be shewn as in the preceding propⁿ that each of the angles CAF, AFC, CFB, FBC is half a right angle, and that AFB is a right angle. The three angles HFK, FKH, KHF of the triangle FHK are equal (I. 32) to two right angles, and FHK is a right angle; therefore the two HFK, FKH are equal to one right angle. But because FB cuts the parallels FH, CB in F, B, the alternate angles CBF, BFH are equal; and CBF is half a right angle: therefore KFH is also half a right angle. Hence the other angle FKH is half a right angle, and equal to the angle KFH; therefore HF is equal (I. 6) to HK. Also the angles KBD, FBC are equal, since they are vertically opposite (I. 15); and FBC is half a right angle: therefore KBD is also half a right angle, and equal to the angle BKD: therefore DB is equal to DK. Now it may be shewn as in the preceding propⁿ that the square of AF is double of the square of AC, and the square of FK double of the square of HF, that is, of CD, because HF is equal to CD, since the opposite sides of parallelograms are equal (I. 34): therefore the squares of AF, FK are double of the squares of AC, CD. But the square of AK is equal (I. 47) to the squares of AF, FK, because AFK is a right angle: therefore the square of AK is double of the squares of AC, CD. Now since DB is equal to DK, the square of DB is equal to the square of DK; to each of these equals add the square of AD: then the squares of AD, DB are equal (Ax. 2) to the squares of AD, DK; but because ADK is a right angle the square of AK is equal to the square of AD, DK; and things that are equal to the same thing are equal to one another (Ax. 1); therefore the square of AK is equal to the squares of AD, DB. And it has been shewn that the square of AK is double of the squares of AC, CD; hence also the squares of AD, DB are double of the squares of AC, CD. Which was to be proved.

PROP. XI. PROB.

To divide a given straight line into two such parts, that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.

Let AB be the given straight line. It is required to divide it into two such parts, that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.



On AB describe (i. 46) the square ACDB; and supposing A to be that extremity of AB adjacent to which the second part is required to be, bisect (i. 10) its side AC in E. Join BE; produce CA to F; from EF cut off (i. 3) EG equal to EB; and on AG describe (i. 46) the square GHKA. Then BAC is a right angle, since it is an angle of the square AD (Def. 30), and CA is produced to F, therefore the adjacent angle FAB is a right angle (Def. 10). But FAK is a right angle, since it is an angle of the square GK; and all right angles are equal (Ax. 11): therefore the angle FAK is equal to the angle FAB; and hence AK falls on AB, and K is a point in AB. Then AB shall be divided in K, so that the rectangle AB, BK is equal to the square of AK.

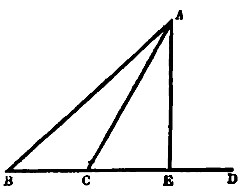
Produce HK, so as to cut CD in L.

Because CA is bisected in E, and produced to G, the rectangle CG, GA together with the square of AE is equal (ii. 6) to the square of EG, that is, to the square of EB, because EG is equal to EB by const^{n}. But because EAB is a right angle, the squares of BA, AE are equal (i. 47) to the square of EB; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the rectangle CG, GA together with the square of AE is equal to the squares of BA, AE. From each of these equals take away the common square of AE; then the remaining rectangle CG, GA is equal (Ax. 3) to the remaining square

of AB. Now, because the opposite sides of squares are parallel, GH, CD are each of them parallel (i. 46, Obs.) to AKB, and therefore to one another (i. 30); and HKL, BD are each of them parallel to GAC, and therefore to one another: hence the figures GL, KD are both parallelograms, and as each contains an angle of a square, and therefore a right angle, they are both rectangles (i. 46, Cor.). And GL is the rectangle CG, GA; for it is contained by CG, GH, and GH is equal to GA, as they are sides of a square. But it was shewn that the rectangle CG, GA is equal to the square of BA; hence GL is equal to ACDB. From each of these equals take away the common part AL; then the remainder GK is equal (Ax. 3) to the remainder KD. But GK is the square of AK; and KD is the rectangle AB, BK, for it is contained by DB, BK, and DB is equal to AB, as they are sides of a square. Therefore the rectangle AB, BK is equal to the square of AK; and the given straight line AB has been divided in K, as was required. Which was to be done.

PROP. XII. THEOR.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle is greater than the squares of the sides including the obtuse angle, by twice the rectangle contained by either of those sides, and the straight line intercepted without the triangle between the obtuse angle and the perpendicular drawn to this side produced from the opposite angular point.



Let ABC be an obtuse-angled triangle, having the obtuse angle ACB; let BC, one of the sides including it, be produced to D; and from the opposite angular point A let AE be drawn (i. 12) perpendicular to BD. Then the square of AB shall be less than the squares of AC, CB by twice the rectangle BC, CE.

Because BE is divided into two parts in C, the square of BE is equal (ii. 4) to the squares of BC, CE together with twice the rectangle BC, CE. To each of these equals

add the square of EA; then the squares of BE, EA are equal (Ax. 2) to the squares BC, CE, EA, together with twice the rectangle BC, CE. But because AEB is a right angle by const^{n}, the square of BA is equal (i. 47) to the squares of BE, EA, and the square of CA is equal to the squares of CE, EA: therefore the square of BA is equal to the squares of BC, CA together with twice the rectangle BC, CE; that is, the square of BA is greater than the squares of BC, CA by twice the rectangle BC, CE. Which was to be proved.

PROP. XIII. THEOR.

In any triangle, the square of the side subtending either of the acute angles is less than the squares of the sides including this angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular drawn to this side, produced if necessary, from the opposite angular point.

Let ABC be a triangle, and let the angle ABC be one of its acute angles; and to BC, one of the sides including the angle ABC, or BC produced to D, let the perpendicular AE be drawn (i. 12) from the opposite angular point A. Then the square of AC shall be less than the squares of AB, BC by twice the rectangle CB, BE.

There are two cases according as the perpendicular AE does not, or does coincide with the side of the triangle AC.

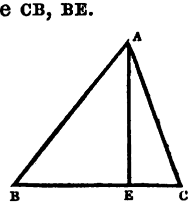


Fig. 1.

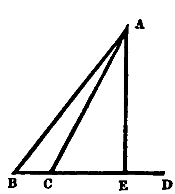


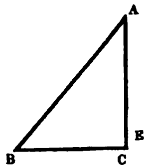
Fig. 2.

I. Let AE not coincide with AC, E either falling in BC (Fig. 1), or in CD without BC (Fig. 2).

Because BC is divided into two parts in E (Fig. 1), or because BE is divided into two parts in C (Fig. 2), therefore in both figures the squares of CB, BE are equal (ii. 7) to twice the rectangle CB, BE together with the square of

CE. To each of these equals add the square of AE; therefore the squares of CB, BE, AE are equal (Ax. 2) to twice the rectangle CB, BE together with the squares of CE, AE. But because AE is at right angles to BC by const^{n}, the square of AB is equal (I. 47) to the squares of BE, EA; and the square of AC is equal to the squares of AE, EC: therefore the squares of CB, BA are equal to twice the rectangle CB, BE together with the square of AC; that is, the square of AC is less than the squares of AB, BC by twice the rectangle CB, BE.

II. Let AE coincide with AC, E and C being one and the same point.

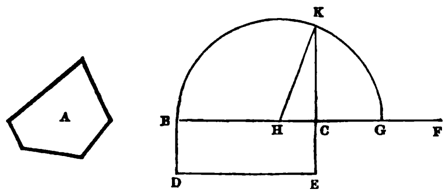


Then, because ACB is a right angle, the square of AB is equal to the squares of AC, CB. To each of these equals add the square of BC; then the squares of AB, BC are equal to the square of AC and twice the square of BC. But the square of BC is the rectangle CB, BE; therefore the squares of AB, BC are equal to twice the rectangle CB, BE, and the square of AC; that is, the square of AC is less than the squares of AB, BC by twice the rectangle CB, BE.

Hence in every case the square of AC is less than the squares of AB, BC by twice the rectangle CB, BE. Which was to be proved.

PROP. XIV. THEOR.

To describe a square that shall be equal to a given polygon.



Let A be the given polygon. It is required to describe a square that shall be equal to A.

I

Take any straight line BC, and to BC apply (i. 45) the parallelogram BDEC equal to A and having one of its angles BCE equal to a right angle. Then BDEC is rectangular; and if two of its adjacent sides BC, CE are equal, all the four will be equal (i. 34), and BE will be a square (Def. 30); that is, what was required is already done. But if BC, CE are not equals; produce BC to F, and from CF cut (i. 3) off CG equal to CE; bisect (i. 10) BG in H; with centre H and radius HB or HG describe a semicircle, and produce EC to cut it in K. Then the square described (i. 46) on CK shall be equal to the polygon A.

Join HK.

Because BG is bisected in H, and divided into two unequal parts in C, the rectangle BC, CG together with the square of CH is equal (ii. 5) to the square of HG, that is, to the square of HK, since HG is equal to HK by the def^n of a circle. But because the angle HCK is adjacent to the right angle BCE, and therefore a right angle (Def. 10), the squares of HC, CK are likewise equal (i. 47) to the square of HK; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the rectangle BC, CG together with the square of CH is equal to the squares of CK, CH. From each of these equals, take away the common square of CH: then the remaining rectangle BC, CG is equal to the remaining square of CK. But BE is the rectangle BC, CG, for it is contained by BC, CE, and CE is equal to CG by const^n: therefore the square of CK is equal to BE. Now by const^n the polygon A is equal to BE; and things that are equal to the same thing are equal to one another: therefore the square described on CK is equal to the given polygon A. Which was to be done.

THE

ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

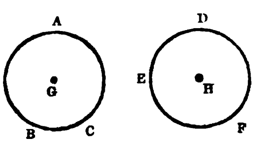
I.

IF the radii of two circles are equal, the circles are equal, and their circumferences are equal; and if two circles are equal, their radii are equal and their circumferences are equal.

OBS. This is neither a defn nor an axiom, but consists of two propns which may be thus proved.

(1) Let ABC, DEF be two circles, of which the centres are G, H, and which have equal radii. Then the circle ABC shall be equal to the circle DEF, and the circumference ABC to the circumference DEF.

For if the circle ABC be applied to the circle DEF, so that the centre G may coincide with the centre H, then every point in the circumference ABC will coincide with a point in the circumference DEF, since the radii of the two circles are equal; that is, the circumference ABC will coincide with the circumference DEF and the circle ABC with the circle DEF. But magnitudes which coincide are equal (Ax. 8): therefore the circumference ABC is equal to the circumference DEF, and the circle ABC to the circle DEF. Which was to be proved.

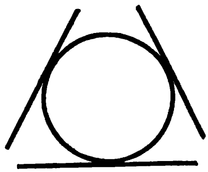


(2) Let ABC, DEF be two equal circles, of which the centres are G, H: then the radii of ABC shall be equal to the radii of DEF, and the circumference ABC to the circumference DEF.

For if the radii of the two circles are unequal, let DEF be that

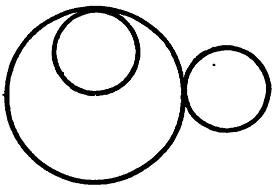
circle, which, if possible, has the larger radii. Apply ABC to DEF as before, then the extremity of each radius of ABC will be nearer to the coincident centres than the extremity of the radius of DEF on which it falls, and therefore the circle ABC will fall entirely within, and be a part of the circle DEF. Therefore the circle ABC is less (Ax. 9) than the circle DEF: which is by hyp[s] impossible. Therefore the radii of the two circles are not unequal: that is, they are equal, and hence by (1), the circumference ABC is equal to the circumference DEF. Which was to be proved.

II.



A straight line is defined to touch a circle at a point, when it meets the circle in that point, and being produced both ways does not cut it.

III.

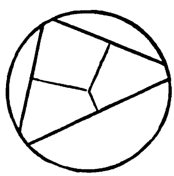


Circles are defined to touch one another at a point, when they meet together at that point, but do not cut another.

IV.

The distance of a straight line from the centre of a circle is the length of the perpendicular drawn to it from the centre.

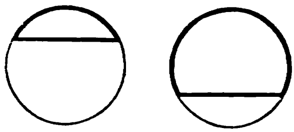
V.



Straight lines are accordingly equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal; and one straight line is said to be farther from or nearer to the centre than another, according as the greater or less perpendicular falls on it.

VI.

A segment of a circle is the figure contained by a part of the circumference of a circle, called an arc of a circle, and the straight line drawn joining the two extremities of the arc.

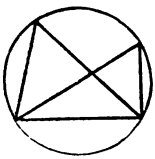


VII.

An angle of a segment is either of the two curvilinear angles contained by the straight line and the arc of the circle.

VIII.

An angle in a segment is any angle contained by two straight lines drawn from any point in the arc of the segment to the extremities of the straight line (which is the base of the segment).

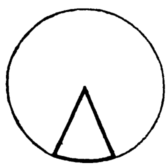


IX.

An angle is said to stand on a part of the circumference, or arc of the circle, when such arc is intercepted by the straight lines that include the angle.

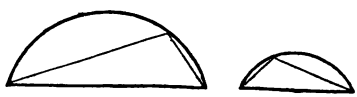
X.

A sector of a circle is the figure contained by two radii, and the arc intercepted between them.



XI.

Segments of circles are defined to be similar, when the angles in them are equal.

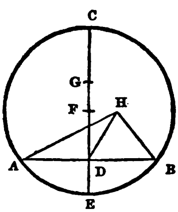


PROPOSITIONS.

PROP. I. THEOR.

To find the centre of a given circle.

Let ABC be the given circle. It is required to find its centre.



Draw any straight line AB, so as to fall entirely within the circle, and have its extremities A, B on the circumference. Bisect (i. 10) AB in D; from D draw (i. 11) a straight line perpendicular to AB; and produce it both ways. Then because AB was taken within the circle, this straight line must meet the circumference in two points; let them be C and E; bisect CE in F. Then F shall be the centre of the circle ABC.

For if F be not the centre, some other point must be the centre, either a point in the straight line CE, or a point without it.

I. Let, if possible, some point G, in the straight line CE, be the centre.

Then because CE is bisected in F by const^{n}, any other point in CE besides F divides CE into two unequal parts; and therefore GE, GC are unequal. But because G is the centre of the circle ABC, GC is equal to GE by def^{n}: which is impossible. Therefore G is not the centre of the circle ABC; and it can be shewn in like manner that no other point in CE but F can be the centre.

II. Let, if possible, some point H, without the straight line CE, be the centre.

Join HA, HD, HB.

Because DA is equal to DB by const^{n}, HA to HB by the def^{n} of a circle, and HD common to the two triangles HAD, HDB; therefore these two triangles have the three sides HA, AD, DH respectively equal to the three sides HB, BD, DH. Therefore they are equal in every respect (i. 8); and hence the angle ADH is equal to the angle BDH. That is, the straight line HD standing on AB makes with it the adjacent angles HDA, HDB equal to one another: therefore by def^{n} each of these angles is a right angle. But FDB is likewise a right angle by const^{n}; and all right angles are equal (Ax. 11); therefore the angle HDB is equal to the angle FDB, that is, the part equal to the whole: which is impossible (Ax. 9). Therefore H is not the centre of the circle ABC; and it can be shewn in like manner that no other point out of CE is the centre.

Hence the point F must be the centre of the circle. Which was to be proved.

COR.—Hence if a straight line be drawn bisecting at right angles a straight line in a circle (i. e. within the circle, and having its extremities on the circumference), and be produced, the centre of the circle shall be in this straight line produced.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle: then the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference. Then the straight line AB, which joins them, shall fall within the circle.

For if it do not fall within, it must either fall without the circle, or partly within and partly without the circle, or else fall on the circumference.

I. Let AB fall, if possible, without the circle (Fig. 1), or partly within and partly without the circle (Fig. 2).

In AB (Fig. 1), or in the part of AB without the circle (Fig. 2), take any point E. Find (iii. 1) D, the centre of the circle; and join DA, DE, DB. Since E was taken without the circle, DE must cut the circumference; let it cut it in the point F.

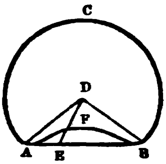


Fig. 1.

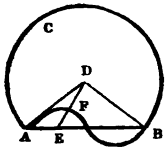
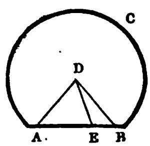


Fig. 2.

Because DA is equal to DB by the def^n of a circle, the angle DAB is equal (i. 5) to the angle DBA; and because the side AE of the triangle DAE is produced to B, the exterior angle DEB is greater (i. 16) than the interior and opposite angle DAB: therefore the angle DEB is likewise greater than the angle DBE. But the greater angle of a triangle is subtended by the greater side (i. 19); therefore DB is greater than DE. But DB is equal to DF by the def^n of a circle; therefore DF is greater than DE, that is, the part greater than the whole: which is impossible (Ax. 9). Therefore AB falls neither without nor partly within and partly without the circle.

II. Let AB fall, if possible, on the circumference.



In AB take any point E; find D, the centre of the circle; and join DA, DE, DB.

It may be shewn exactly as in the former case, that DB is greater than DE. But DB is equal to DE by the def^n of the circle: which is impossible. Therefore AB does not fall on the circumference.

Hence the straight line AB, joining A and B, can only fall within the circle. Which was to be proved.

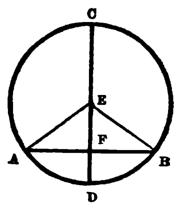
PROP. III. THEOR.

If a straight line drawn through the centre of a circle bisect a straight line in the circle which does not pass through the centre: then it shall cut it at right angles. And if it cuts it at right angles: then it shall bisect it.

Let ABC be a circle; CD a straight line drawn through the centre; AB a straight line in the circle.

I. Let CD bisect AB in F. Then it shall cut it at right angles.

Bisect (i. 10) CD in E, which will be the centre of the circle; and join EA, EB.



Because EA is equal to EB by the defⁿ of a circle, AF to FB by hypˢ, and EF common to the two triangles EAF, EBF; therefore these two triangles have the three sides EA, AF, FE respectively equal to the three sides EB, BF, FE. Therefore they are equal in every respect (i. 8), and hence the angle AFE is equal to the angle BFE. That is, the straight line EF, standing on AB, makes with it the adjacent angles AFE, BFE equal to one another: therefore by defⁿ each of these angles is a right angle, and CD is at right angles to AB. Which was to be proved.

II. Let CD cut AB at right angles in F. Then it shall bisect it.

Construct as before.

Because EA is equal to EB by the defⁿ of a circle, the angle EAB is equal (i. 5) to the angle EBA; and the angles AFE, BFE are equals (Def. 10), since CD is by hypˢ, at right angles to AB: therefore the two triangles EAF, EBF have the two angles EAF, EFA of the one respectively equal to the two angles EBF, EFB of the other, and the sides EA, EB opposite to the equal angles in each equal. Therefore these two triangles are equal in every respect (i. 26); and hence the side AF is equal to the side BF. That is, CD bisects AB in F. Which was to be proved.

PROP. IV. THEOR.

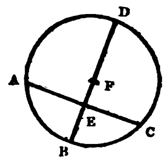
If in a circle two straight lines cut one another which do not both pass through the centre: then they shall not bisect each other.

Let ABC be a circle, and AC, BD two straight lines in the circle, which cut one another in E, and which do not

both pass through the centre. Then they shall not bisect one another; that is, E shall not be the middle point both of AC and of BD.

Since by hyp[s] both the straight lines do not pass through the centre, there are two cases, according as one passes through the centre and the other does not, or as neither passes through the centre.

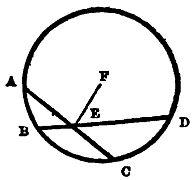
I. Let one of the straight lines BD pass through the centre, and the other not.



Bisect (i. 10) BD in F; then F will be the centre.

Since FB is equal to FD by the def[n] of a circle, it is clear that E cannot be the middle point of BD, or that BD cannot be bisected in E by AC, which does not pass through the centre.

II. Let neither of the straight lines AC, BD pass through the centre.



If they do bisect one another: let, if possible, both AE be equal to EC, and BE to ED.

Find (iii. 1) the centre F of the circle, and join FE.

Because FE, drawn through the centre, bisects the straight line AC in the circle, which does not pass through the centre, it cuts it at right angles (iii. 3); and hence FEA is a right angle. Again, because FE, drawn through the centre, bisects the straight line BD in the circle, which does not pass through the centre, it cuts it at right angles, and hence FEB is likewise a right angle. And all right angles are equal (Ax. 11): therefore the angle FEA is equal to the angle FEB, that is, the part equal to the whole: which is impossible (Ax. 9). Therefore AE is not equal to EC at the same time that BE is equal to ED.

Hence AC, BD do not bisect one another in E. Which was to be proved.

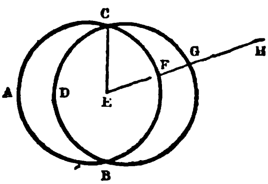
PROP. V. THEOR.

If two circles cut one another: then they shall not have the same centre.

Let the two circles ABC, CDG cut one another. Then they shall not have the same centre.

For if they have: let them, if possible, have the same centre E.

Join E with C, C being a point in which they cut one another; and through E draw any straight line EFGH so as to cut the circle ABC in the point F and the circle CDG in another point G.



Because E is the centre of the circle ABC, EF is equal to EC by def^n, and because E is the centre of the circle CDG, EG is equal to EC by def^n; and things that are equal to the same thing are equal to one another (Ax. 1): therefore EF is equal to EG, that is, the part equal to the whole: which is impossible (Ax. 9). Therefore E is not the centre of both the circles; and the same may in like manner be shewn of every other point: hence the circles ABC, CDG do not have the same centre. Which was to be proved.

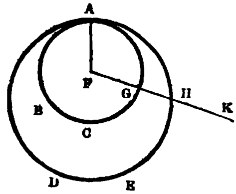
PROP. VI. THEOR.

If one circle touch another circle on the inside: then they shall not have the same centre.

Let the circle ABC touch the circle ADE on the inside at a point A. Then they shall not have the same centre.

For if they do: let them, if possible, have the same centre F.

Join F with A; and through F draw any straight line FGHK so as to cut the circle ABC in the point G, and the circle ADE in another point H.



Because F is the centre of the circle ABC, FG is equal to FA by

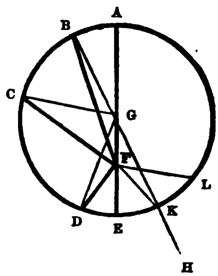
def^n, and because F is the centre of the circle ADE, FH is equal to FA by def^n; and things that are equal to the same thing are equal to one another (Ax. 1): therefore FG is equal to FH, that is, the part equal to the whole: which is impossible (Ax. 9). Therefore F is not the centre of both the circles; and the same may be shewn in like manner of every other point: hence the circles ABC, ADE do not have the same centre. Which was to be proved.

PROP. VII. THEOR.

If any point be taken within a circle, which is not the centre: then

(1) of all the straight lines which can be drawn from it to the circumference, the greatest shall be that which passes through the centre; and of the rest that which is nearer to the greatest shall be always greater than one more remote; and the least shall be the other part of the diameter through the point;

(2) from this point there can be drawn one and only one straight line to the circumference equal to a given straight line drawn from it to the circumference, which shall be on the opposite side of the diameter through it.



Let ABC be a circle, and let F be any point within it which is not the centre. Then:—

I. Of all the straight lines FA, FB, FC, FD, FE that can be drawn from F to the circumference, FA which passes through the centre G shall be the greatest; of the rest FB which is nearer to FA than FC shall be greater than FC; FC which is nearer to FA than FD shall be greater than FD; and FD which is the other part of the diameter AGFE through F shall be the least.

Join GB, GC, GD.

Then by the def^n of a circle, GA, GB, GC, GD, GE are all

equal. Now because GA is equal to GB, to each of these equals add GF; then the whole AF is equal (Ax. 2) to BG, GF. But the two sides BG, GF of the triangle BGF are greater than the third BF (I. 20); therefore AF is greater than BF. Again, because BG is equal to CG and GF common to the two triangles BGF, CGF, therefore these two triangles have the two sides BG, GF respectively equal to the two sides CG, GF, and the included angle BGF is greater than the included angle CGF. Therefore the base BF of the one which has the greater included angle is greater (I. 24) than the base CF: and by like reasoning it may be shewn that FC is greater than FD. Lastly, because the two sides GF, FD of the triangle GFD are greater (I. 20) than the third GD, and GD is equal to GE; therefore GF, FD are greater than GD. From each of these unequals take away the common part GF: then the remainder FD is greater (Ax. 5) than the remainder FE. Hence FA is the greatest, FB greater than FC, FC greater than FD, and FE the least. Which was to be proved.

II. From F there can be drawn one and only one straight line to the circumference equal to a given straight line FD drawn from F to the circumference, which will lie on the opposite side of the diameter AGFE to FD.

Join GD; and at the point G in the straight line EG make (I. 23) the angle FGH equal to the angle FGD. Let GH cut the circle in K, and join FK. Then FK shall be equal to FD.

Because GD is equal to GK by the def^{n} of a circle, GF common to the two triangles, GDE, GKF, and the angles FGD, FGK equal by const^{n}; therefore these two triangles have the two sides DG, GF respectively equal to the two sides KG, GF, and the included angle DGF equal to the included angle KGF. Therefore they are equal in every respect (I. 4); and hence the base FD is equal to the base FK. Also besides FK no other straight line can be drawn from F to the circumference equal to FD: for if there can, let it be FL. Then because FL is equal to FD, and FK to FD, and things that are equal to the same thing are equal to one another (Ax. 1): therefore FL is equal to FK, that is, the straight line drawn from F to the circumference which is nearer to FGA, which passes through the centre

K

is equal to one which is more remote: which by Part I. of the prop^n is impossible. Hence from F one line and one only FK can be drawn to the circumference equal to FD, lying on the opposite side of the diameter through F. Which was to be proved.

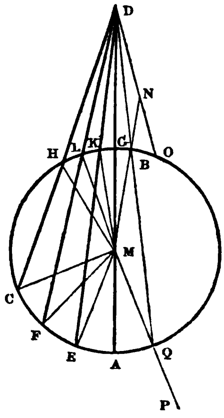
PROP. VIII. THEOR.

If any point be taken without a circle: then

(1) of all straight lines drawn from it to the concave part of the circumference, the greatest shall be that which passes through the centre, and of the rest that which is nearer to the greatest shall be always greater than one more remote;

(2) of all straight lines drawn from it to the convex part of the circumference, the least shall be that one which if produced passes through the centre, and of the rest that which is nearer to the least shall always be less than one more remote;

(3) from this point there can be drawn one and one only straight line to the circumference equal to a given straight line drawn from it to the circumference, which shall lie on the opposite side of the straight line joining the point and the centre of the circle.



Let ABC be a circle, and let D be any point without it. Then:—

I. Of all the straight lines DA, DE, DF, DC that can be drawn from D to the concave part of the circumference AEFC, DA which passes through the centre of the circle M shall be the greatest; and of the rest DE which is nearer to DA than DF shall be greater than DF, and DF which is nearer to DA than DC shall be greater than DC.

Join ME, MF, MC.

Then by the def^n of a circle MA, ME, MF, MC are all equal. Now because MA is equal to ME, to each

of these equals add MD: then the whole DA is equal (Ax. 2) to EM, MD. But the two sides EM, MD of the triangle EMD are greater (i. 20) than the third DE: therefore DA is greater than DE. Again, because ME is equal to MF, and MD common to the two triangles EMD, FMD: therefore these two triangles have the two sides EM, MD respectively equal to the two sides FM, MD, but the included angle EMD is greater than the included angle FMD. Therefore the base DE of the one which has the greater included angle is greater (i. 24) than the base DF; and by like reasoning it may be shewn that DF is greater than DC. Hence DA is the greatest, DE greater than DF, and DF greater than DC. Which was to be proved.

II. Of all the straight lines DG, DK, DL, DH that can be drawn from D to the convex part of the circumference HLKG, DG, which if produced to A, passes through the centre M, shall be the least; and of the rest, DK which is nearer to DG than DL shall be less than DL, and DN which is nearer to DG than DH shall be less than DH.

Join MK, ML, MH.

Then by the def^n of a circle, MG, MK, ML, MH are all equal. Now the two sides MK, KD of the triangle MKD are greater (i. 20) than the third DM; from each of these unequals take away the equals MK, MG respectively: then the remainder DK is greater (Ax. 5) than the remainder DG. Again, because MK, DK are drawn from the extremities M, D of the side MD the triangle DML to a point K within it, MK, KD are together less than ML, LD; from these unequals take away the equals MK, ML respectively: then the remainder DK is less than the remainder DL: and by like reasoning it may be shewn that DL is less than DH. Hence DG is the least, DK less than DL, and DL less than DH. Which was to be proved.

III. From D there can be drawn one and one only straight line to the circumference equal to a given straight line drawn from D to the circumference which shall lie on the opposite side of the straight line joining D and the centre of the circle M.

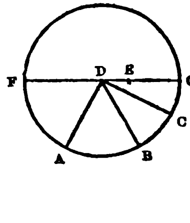
Suppose first that the given straight line DK is drawn to the convex part of the circumference. Join MK; and at the point M in the straight line DM make (i. 23) the

angle DMN equal to the angle DMK. Let B be the point where MN cuts the circumference, and join DB. Then DB shall be equal to DK.

By the defn of a circle MK is equal to MB, and MD is common to the two triangles DMK, DMB: therefore these two triangles have the two sides KM, MD equal to the two sides BM, MD respectively, and the included angle KMD is equal to the included angle BMD by constn. Therefore they are equal in every respect (I. 4); and hence the base DB is equal to the base DK. Also besides DB no other straight line can be drawn from D to the circumference equal to DK: for if there can let it be DO. Then because DO is equal to DK, and DB is equal to DK; and things that are equal to the same thing are equal to one another (Ax. 1): therefore DO is equal to DB, that is, the straight line DB drawn from D to the convex part of the circumference, which is nearer to DM, is equal to the one DO which is more remote: which by Part II. of the propn is impossible. Hence from D one and only straight line DB can be drawn to the circumference equal to DK, lying on the opposite side of DM. And if the given straight line had been drawn to the concave part of the circumference as DE, it might by joining ME, making the angle DMP equal to DME, and joining Q where MP cuts the circumference with D be shewn in like manner that one and only straight line DQ could be drawn to the circumference equal to DE. Which was to be proved.

PROP. IX. THEOR.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference: then that point shall be the centre of the circle.



Let the point D be taken within the circle ABC, from which there are drawn more than two equal straight lines to the circumference. Then D shall be the centre of the circle.

For if it be not: let if possible some other point E be the centre.

Join DE, and produce it both ways to cut the circumference in F, G.

Of the equal straight lines drawn from D to the circumference, which by hyp[s] are more than two, let DA, DB, DC be three; and because D is a point within the circle, not the centre, and DG, DC, DB, DA are straight lines drawn from it to the circumference, DG which passes through the centre E is the greatest, DC which is nearer to DG than DB is greater than DB, and DB which is nearer to DG than DA is greater than DA (iii. 7). But by hyp[s], DC, DB, DA are all equal: which is impossible. Therefore E is not the centre of the circle ABC; and in like manner it may be shewn that no other point but D is the centre. Hence D is the centre of the circle. Which was to be proved.

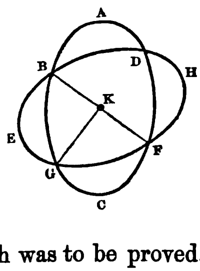
PROP. X. THEOR.

One circle cannot cut another circle in more than two points.

For if it can: let, if possible, the circle ABC cut the circle DEF in more than two points.

Of these points of intersection, which are supposed more than two, let B, G, F be three; find (iii. 1) the centre K of the circle ABC, and join KB, KG, KF.

Because K is the centre of the circle ABC, the straight lines KB, KG, KF are equal by def[n]. But because from the point K within the circle DEF more than two equal straight lines KB, KG, KF are drawn to the circumference, the point K is the centre (iii. 9) of the circle DEF. But K is also the centre of the circle ABC; hence the two circles ABC, DEF which cut one another have the same centre K: which is impossible (iii. 5). Therefore one circle cannot cut another in more than two points. Which was to be proved.

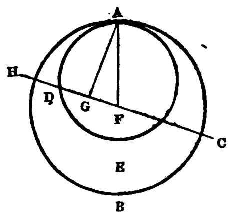


PROP. XI. THEOR.

If one circle touch another on the inside: then the

straight line joining their centres shall, being produced, pass through the point of contact.

Let the circle ADE touch the circle BAC on the inside at the point A. Then the straight line which joins their centres, being produced, shall pass through the point of contact A.



For if it does not: let it fall, if possible, otherwise as CFGDH; F being the centre of the circle ABC; G the centre of ADE; H, C the points where FG produced cuts the circle ABC; and D the point where FH cuts the circle ADE.

Join AF, AG.

Because the two sides AG, GF of the triangle AGF are greater (i. 20) than the third AF; and AF is equal to FH by def^n, since F is the centre of the circle ABC: therefore AG, GF are greater than FH. From each of these unequals take away the common part FG: then the remainder AG is greater (Ax. 5) than the remainder GH. But because G is the centre of the circle ADE, AG is equal to GD by def^n; therefore GD is greater than GH, that is, the part greater than the whole: which is impossible (Ax. 9). Therefore the straight line which joins the centres of the circles, being produced, cannot fall otherwise than on the point of contact A, that is, it passes through it. Which was to be proved.

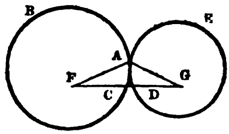
PROP. XII. THEOR.

If two circles touch one another externally: then the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch each other externally at the point A. Then the straight line which joins their centres shall pass through the point of contact A.

For if it does not: let it fall, if possible, otherwise as FCDG; F, G being the centres of the two circles, and C, D the points where FG cuts them.

Join FA, GA.



By the def^{n} of a circle, since F is the centre of the circle BAC, and G the centre of the circle EAD, FA is equal to FC, and GA to GD. Therefore adding equals to equals, FA, AG are equal (Ax. 2) to FC, DG; but the whole FG is greater (Ax. 9) than its part FC, DG: hence FG is likewise greater than FA, AG; that is, the two sides FA, AG of the triangle FAG are less than the third FG: which is impossible (i. 20). Hence the straight line which joins the centres of the two circles cannot pass otherwise than through the point of contact A. Which was to be proved.

PROP. XIII. THEOR.

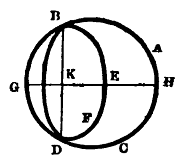
One circle cannot touch another at more points than one, whether it touches it on the inside or externally.

There are two cases according as the circle touches the other on the inside or externally.

I. One circle cannot touch another on the inside at more points than one.

For if it can: let, if possible, the circle EBF touch the circle ABC on the inside at more points than one.

Of the points of contact, which are supposed more than one, let B, D be two. Join BD; bisect (i. 10) BD in K, and through K draw (i. 11) GKH at right angles to BD, cutting the exterior circle in G, H.

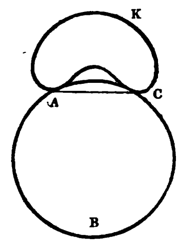


Because B, D are in the circumference of the circle ABC, the straight line BD which joins them falls within (iii. 2) the circle ABC; and GH bisects BD at right angles: therefore the centre of the circle ABC lies (iii. 1, Cor.) in GH. Similarly it may be shewn that the centre of the circle EBF lies in GH: therefore GH is the straight line, which joins their centres, produced, and hence GH must pass through (iii. 11) the

point of contact; but it does not pass through either B or D, because they are each without the straight line GH: which is absurd. Therefore one circle cannot touch another on the inside at more points than one. Which was to be proved.

II. One circle cannot touch another externally at more points than one.

For if it can: let, if possible, the circle ACK touch the circle ABC at more points than one.



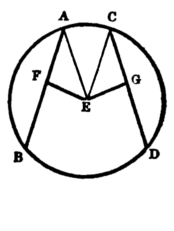
Of these points of contact, which are supposed more than one, let A, C be two; and join AC.

Because A, C are in the circumference of the circle ACK, the straight line AC which joins them falls within (iii. 2) the circle AKC; and the circle ACK is without the circle ABC by hyp^s: therefore AC falls without the circle ABC. But because A, C are in the circumference of the circle ABC, the straight line AC which joins them falls within the circle ABC: which is impossible. Therefore one circle cannot touch another externally in more points than one. Which was to be proved.

PROP. XIV. THEOR.

Straight lines in a circle that are equal to one another shall be equally distant from the centre. And those that are equally distant from the centre shall be equal to one another.

Let ABC be a circle, of which the centre is E; AB, CD straight lines in it; EF, EG perpendiculars drawn to them from E, and the lengths of which are by def^n (iii. Def. 4) the distances of AB, CD from the centre. Then:—



I. If AB, CD be equal to one another, they shall be equally distant from the centre, that is, EF, EG shall be equal.

Join EA, EC.

Because EF drawn through the centre cuts the straight line in the circle AB, which does not pass through the centre, at right angles, it also bisects (iii. 3) it; and hence AF is equal to FB, and AB double of AF. For the same reason CD is double of CG. Now because EA is equal to EC by the def^{n} of a circle, the square of EA is equal to the square of EC; but the squares of AF, FE are equal (i. 47) to the square of AE, since AFE is a right angle, and the squares of EG, GC are equal to the square of EC, since EGC is a right angle: therefore the squares of AF, FE are equal to the squares of CG, GE. But the square of AF is equal to the square of CG, because AF is equal to CG, since they are the halves of AB, CD, and the halves of equal things are equal (Ax. 7): hence taking away equals from equals, the remaining square of FE is equal (Ax. 3) to the remaining square of EG, and therefore FE is equal to EG. Which was to be proved.

II. If AB, CD be equally distant from the centre, that is, if EF be equal to EG, they shall be equal.

It may be shewn as before that AB is double of AF and CD double of CG, and the squares of EF, FA are equal to the squares of EG, GC. But the square of EF is equal to the square of EG, because EF is equal to EG by hyp^{s}: hence taking away equals from equals, the remaining square of AF is equal (Ax. 3) to the remaining square of CG, and therefore AF is equal to CG. But AB is double of AF, and CD double of CG, and the doubles of equal things are equal (Ax. 6): hence AB is equal to CD. Which was to be proved.

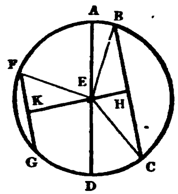
PROP. XV. THEOR.

Of straight lines in a circle, the diameter shall be the greatest; and of the rest, that which is nearer to the centre shall be always greater than one more remote, and the greater shall always be nearer to the centre than the less.

Let ABC be a circle, of which the centre is E; AED any diameter; BC, FG straight lines in the circle; EH, EK per-

pendiculars drawn to them from E, and the lengths of which are by defⁿ (iii. Def. 4) the distances of BC, FG from the centre. Then:—

I. The diameter AD shall be greater than any straight line BC which is not a diameter; and of the rest, if BC be nearer to the centre than FG, that is, if EH be less than EK, BC shall be greater than FG.



Join EB, EC, EF.

Then EB, EC, EF, EA, ED are all equal by the defⁿ of a circle. And because EA is equal to EB, and ED to EC; therefore, adding equals to equals, AE, ED, that is, AD is equal (Ax. 2) to EB, EC. But the two sides EB, EC of the triangle BEC are greater (i. 20) than the third BC; therefore also AD is greater than BC. Again, it may be shewn, as in the preceding propⁿ, that BC is double of BH, and FG double of FK, and that the squares of EH, HB are equal to the squares of EK, KF. Now the square of EH is less than the square of EK, because EH is less than EK by hypˢ: hence the remaining square of BH is greater than the square of FK, and therefore BH greater than FK. Hence also, since BC, FG are the doubles of BH, FK, BC is greater than FG. Therefore the diameter BD is greater than any other straight line in the circle BC, and BC, the nearer to the centre, is greater than FG, the more remote. Which was to be proved.

II. If BC be greater than FG, BC shall be nearer to the centre than FG, that is, EH shall be less than EK.

Construct as before.

Then it may be shewn, as before, that BC is double of BH, and FG of FK, and the squares of BH, HE are equal to the squares of EK, KF. Now, because BC is greater than FG by hypˢ, therefore BH, the half of BC, is greater than FK, the half of FG, and the square of BH greater than the square of FK: hence the remaining square of HE is less than the remaining square of EK, and therefore HE less than EK; that is, BC the greater straight line is

nearer to the centre than FG the less. Which was to be proved.

PROP. XVI. THEOR.

If a straight line be drawn at right angles to any radius of a circle from its extremity: then

(1) this straight line shall fall without the circle;

(2) no straight line can be drawn from the extremity of the radius between this straight line and the radius, so as not to cut the circle.

Let ABC be a circle, of which D is the centre and DA any radius; and let the straight line AE be drawn at right angles to DA, from its extremity A. Then :—

I. AE shall fall without the circle ABC.

For if it do not fall without, it must either fall wholly or partly within the circle, or else on the circumference.

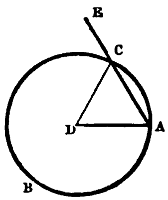


Fig. 1.

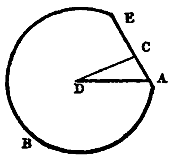


Fig. 2.

Firstly, let AE fall, if possible, wholly or partly within the circle (Fig. 1), and since it must, produced if necessary, cut the circumference, let C be the point where it does; and join DC. Then because DA is equal to DC by the def^{n} of a circle, the angle DAC is equal (I. 5) to the angle DCA; but DAC is a right angle by hyp^{s}; therefore DCA is a right angle; and hence the two angles DAC, DCA, of the triangle DAC, are equal to two right angles: which is impossible (I. 17). Therefore AE cannot fall wholly or partly within the circumference.

Secondly, let AE fall, if possible, on the circumference (Fig. 2). In AE take any point C; and join DC. Then it may be shewn, as before, that the two angles DAC, DCA, of the triangle DAC, are equal to two right angles: which is impossible. Therefore AE cannot fall on the circumference.

Hence AE can only fall without the circle. Which was to be proved.

II. No straight line can be drawn from A, between AE and the radius AD, which shall not cut the circle.

For if there can be drawn such a line, let it, if possible, be AF: then since AF does not cut the circle, it must either fall without the circle or on the circumference.

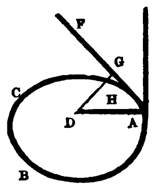


Fig. 1.

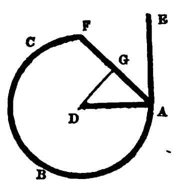


Fig. 2.

Firstly, let AF fall without the circle (Fig. 1). From D draw (i. 12) DG perpendicular to AF; and since AF falls without the circle, DG must cut the circumference in some point; let this point be H. Then because AF is drawn between AE and AD, the angle DAG is less (Ax. 9) than the angle DAE; but DAE is a right angle by hyp^s: therefore DAG is less than a right angle. But DGA is a right angle by const^n: therefore the angle DGA is greater than the angle DAG; and the greater angle of a triangle is subtended by the greater side (i. 19): therefore DA is greater than DG. Now DA is equal to DH by the def^n of a circle; hence DH is greater than DG, that is, the part greater than the whole: which is impossible. Therefore AF cannot fall without the circle.

Secondly, let AF fall, if possible, on the circumference (Fig. 2). From D draw DG at right angles to AF, G being in AF, and therefore also in the circumference of the circle. Then, as before, it may be shewn that DA is greater than DG. But by the def^n of a circle DA is equal to DG: which is impossible. Therefore AF cannot fall on the circumference.

Hence since no straight line can be drawn between AE and the radius so as either to fall without the circle or on the circumference, all such straight lines must cut the circle. Which was to be proved.

COR.—If a straight line be drawn at right angles to any radius of a circle from its extremity, it shall

touch the circle at the extremity of the radius. And a straight line touching the circle at one point shall touch it at no other point.

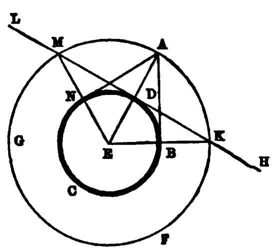
The line drawn at right angles to the radius from its extremity meets the circle in this point, and if produced either way cannot cut it; for if so, two of its points would be in the circumference, and part of it would fall within the circle (iii. 2): which by Part I. of the propⁿ is impossible. Hence this line touches (iii. Def. 2) the circle at the extremity of the radius. Also a line touching the circle at one point, can touch it at no other: for if so, two of its points would be in the circumference, and part of it would fall within the circle, and it would therefore cut it: which is impossible. Which was to be proved.

PROP. XVII. PROB.

From a given point either without the circle or in the circumference to draw a straight line, which shall touch a given circle.

Let A be the given point; BCD the given circle. It is required to draw from A a line which shall touch the circle.

I. Let A be a point without the circle BCD.



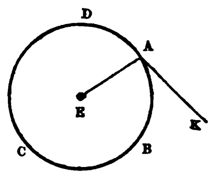
Find (iii. 1) the centre E; join EA cutting the circle in D. With centre E and radius EA describe the circle AFG. From D draw (i. 11) a straight line at right angles to AE, and produce it both ways to H, L, cutting the circle AFG in K and M. Then, if the touching line be required to be drawn on the side of AE towards K, join EK cutting the circle BCD in B: and join AB. Then AB shall touch the circle BCD at B.

By the defⁿ of a circle, EA is equal to EK and ED to EB, and the angle AEK is common to the two triangles AEB, KED; hence they have the two sides AE, EB respectively equal to the two sides KE, ED, and the included angle AEB equal to the included angle KED. Therefore they are equal

in every respect (i. 4); and hence the angle ABE is equal to the angle KDE and therefore a right angle, since EDK is one by constr: that is BA is perpendicular to the radius EB of the given circle BCD from its extremity B, and therefore AB touches (iii. 16. Cor.) it at B. Had the touching line been required to be drawn on the side of AE towards M, it might be shewn by joining EM cutting the circle BCD in N, and joining AN that AN touches the circle BCD at N. Which was to be done.

II. Let A be a point in the circumference BCD.

Find the centre E; join EA; and from A draw AK perpendicular to EA. Then AK shall touch the circle BCD at A.

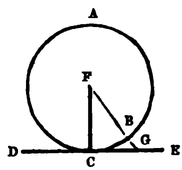


Since AK is perpendicular to the radius EA of the given circle from its extremity A, AK touches it at the given point A. Which was to be done.

PROP. XVIII. THEOR.

If a straight line touch a circle: then the straight line drawn from the centre to the point of contact shall be perpendicular to the touching line.

Let DE touch the circle ABC, and let the point of contact C and F the centre be joined. Then FC shall be perpendicular to DE.



For if not: then some other straight line than FC, drawn from F, will be perpendicular to DE; let it, if possible, be FG. Since DE falls without the circle, FG must cut the circle in some point; let it do so in B.

The two angles FCG, FGC of the triangle FCG are less than two right angles (i. 17); and FGC is supposed one: hence the other FCG is less than a right angle, and therefore than FGC. But the greater angle of a triangle is subtended by the greater side (i. 19): therefore FC is greater than FG. But FC is equal to FB by the defn of a circle; therefore FB is greater than FG, that is, the part greater than the whole: which is impossible (Ax. 9). Therefore FG is not perpendicular to DE; and by like rea-

soning no other straight line but FC drawn from F is perpendicular to DE: hence FC is perpendicular to DE. Which was to be proved.

OBS. From a point without a circle two and two only lines can be drawn touching it, one on each side of the line joining the point with the centre, which shall be equal. And through a point in the circumference one line only can be drawn touching the circle.

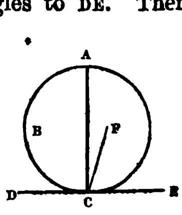
(1) From A without the circle BCD (see Fig. Prop. 17, Part i.) two lines AB, AN can be drawn touching it, one on each side of AE which joins A and the centre E. Now (iii. 17) ABE, ANE are right angles: hence the squares of AB, BE and those of AN, NE are equal to the square of AE (i. 47), and therefore to one another (Ax. 1). But the square of EB is equal to that of NE, since EB is equal to NE; hence taking away equals from equals (Ax. 3), the square of AB is equal to that of AM, and AB to AN. Neither can any other line but AB, AN be drawn from A touching the circle: for if so, let it fall between AB and AE. Then joining the centre with the point of contact, the angle included by this line and the radius is a right angle by prop^n, and therefore equal to the angle ABE: which is impossible (i. 21).

(2) Also through a point in the circumference only one line can be drawn touching the circle; for if there can be another, join the centre with the point of contact, then the angles made by the radius and each of the touching lines are right angles, and therefore equal: which is impossible (Ax. 9).

PROP. XIX. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to it: then the centre of the circle shall be in that line.

Let DE touch the circle ABC, and from the point of contact C let CA be drawn at right angles to DE. Then the centre of the circle shall be in CA.



For if it is not: let it, if possible, be without the straight line CA, and be some point F. Join CF.

Because DE touches the circle ACE and FC is drawn from the centre F to the point of contact C, FC is perpendicular (iii. 18) to DE, and FCE is a right angle. But ACE is a right angle by hyp^s; and all right angles are equal (Ax. 11); therefore FCE is equal to ACE, that is, the part equal to the whole: which is im-

possible (Ax. 6). Therefore F is not the centre of the circle ABC; and in the same manner it may be shewn that no other point without CA is the centre; that is, the centre is in CA. Which was to be proved.

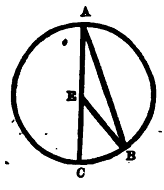
PROP. XX. THEOR.

The angle at the centre of a circle shall be double of the angle at the circumference on the same base, that is, part of the circumference or arc of the circle.

Let ABC be a circle; the centre is E; BEC, BAC angles at the centre and circumference, having the same arc BC for its base. Then the angle BEC shall be double of the angle BAC.

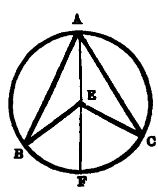
There are three cases according as the centre E is in one of the straight lines BA, AC including the angle BAC, or is within the angle BAC, or is without it.

I. Let E be in AC one of the straight lines AB, AC.



Since EA is equal to EB by the def[n] of a circle, the angle EAB is equal (i. 5) to the angle EBA, and the angles EAB, EBA double of the angle EAB. But since the side AE of the triangle BAE is produced to C, the exterior angle BEC is equal (i. 32) to the angles EAB, EBA: therefore the angle BEC is double of the angle BAC.

II. Let E be within the angle BAC.



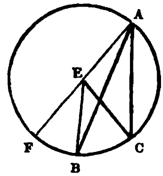
Join AE, and produce it to meet the circumference in F.

By Case I. the angle BEF at the centre is double of the angle BAF at the circumference on the same base BF, and the angle FEC at the centre is double of the angle FAC at the circumference on the same base FC. Therefore the whole angle BEC is double of the whole angle BAC.

III. Let E be without the angle BAC.

Construct as in Case II.

Then as in that case, the angle FEC is double of the



angle FAC, and the angle FEB double of the angle FAB. Therefore the remaining angle BEC is double of the remaining angle BAC.

Hence in every case the angle BEC at the centre is double of the angle BAC at the circumference on the same arc BC. Which was to be proved.

PROP. XXI. THEOR.

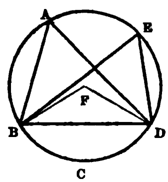
The angles in the same segment of a circle shall be equal to one another.

Let ABC be a circle, and BAD, BED angles in the same segment BAED. Then the angles BAD, BED shall be equal.

There are two cases according as the segment BAED is greater, or not greater than a semicircle.

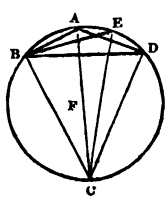
I. Let the segment BAED be greater than a semicircle.

Find (iii. 1) the centre F of the circle; and join BF, DF.



Because the angle BFD is at the centre, and the angle BAD at the circumference has the same arc BCD for its base, therefore the angle BFD is double of the angle BAD. For the same reason the angle BFD is double of the angle BED; and things that are halves of the same thing are equal to one another (Ax. 7): therefore the angle BAD is equal to the angle BED.

II. Let the segment BAED be not greater than a semicircle.



Find the centre F of the circle; join AF; produce AF to meet the circumference in C; and join CE, CB, CD.

Because AC is a diameter by constⁿ, the segment AEDC is a semicircle: therefore each of the segments BAEDC, DEABC is greater than

a semicircle. Now because BAEDC is a segment greater than a semicircle, the angles in it BAC, BEC are equal by the first case; and because DEABC is a segment greater than a semicircle, the angles in it CAD, CED are equal for the same reason. Hence adding equals to equals, the whole angle BAD is equal (Ax. 2) to the whole angle BED.

Hence in every case the angles BAD, BED in the same segment BAED are equal. Which was to be proved.

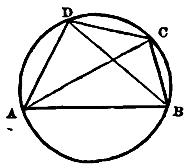
PROP. XXII. THEOR.

The opposite angles of any quadrilateral figure inscribed in a circle shall be together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABC. Then each pair of its opposite angles shall be equal to two right angles.

I. The opposite angles ABC, ADC shall be equal to two right angles.

Join AC, BD.



Then the three angles of the triangle CAB, viz. CAB, ABC, BCA, are equal (i. 32) to two right angles. Now the angle CAB is equal (iii. 21) to the angle CDB, because they are in the same segment BADC; and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB: hence adding equals to equals, the whole angle ADC is equal (Ax. 2) to the angles CAB, BCA. To each of these equals add the angle ABC; then the angles ADC, ABC are equal to the three angles CAB, ABC, BCA. But these three angles have been shewn to be equal to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles ADC, ABC are equal to two right angles.

II. The opposite angles BAD, BCD shall be equal to two right angles.

For by taking the three angles of the triangle BAD, viz.

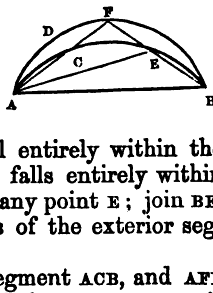
BAD, ADB, DBA, and proceeding in the same manner as before, it may be shewn that the angles BAD, BCD are equal to two right angles.

Hence each pair of the opposite angles, viz. ABC, ADC and BAD, BCD are equal to two right angles. Which was to be proved.

PROP. XXIII. THEOR.

On the same straight line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

For if there can: let there, if possible, on the same straight line AB and on the same side of it be two similar segments of circles, ACB, ADB, not coinciding with one another.



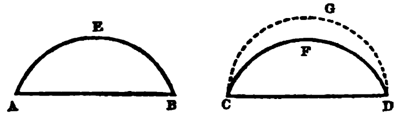
Then, because the circle of which ACB is a segment cuts the circle of which ADB is a segment in the two points A, B, these two circles cannot cut one another in any more points (iii. 10); and therefore one of the segments must fall entirely within the other. Let ACB be the one which falls entirely within the other ADB; in the arc ACB take any point E; join BE, and produce it to meet the arc ADB of the exterior segment in F. Join AE, AF.

Because AEB is an angle in the segment ACB, and AFB is an angle in the segment ADB; and the segment ACB is similar to the segment ADB by hyp^s: therefore by the def^n of similar segments (iii. Def. 11), the angle AEB is equal to the angle AFB. But because the side FE of the triangle AFE is produced to D, the exterior angle AEB is greater than the interior and opposite angle AFE (i. 16); and it has just been shewn to be equal to it: which is impossible. Therefore on the same straight line and on the same side of it, there cannot be two similar segments of a circle, not coinciding with one another. Which was to be proved.

PROP. XXIV. THEOR.

Similar segments of circles on equal straight lines shall be equal to one another.

Let AEB, CFD be similar segments of circles on the equal straight lines AB, CD. Then the segment AEB shall be equal to the segment CFD.



Let the segment AEB be applied to the segment CFD so that the point A may be in C, and the straight line AB on CD, and the arcs of the segments on the same side of CD.

Then the point B shall coincide with the point D, because AB is equal to CD by hyp[s]. Therefore the straight line AB coinciding with CD, and the segments falling on the same side of CD, the segment AEB must coincide with the segment CFD; because if it did not, the arc AEB would take some other direction as CGB, and on the same straight line CD, and on the same side of it, there would be two similar segments of circles CGD, CFD not coinciding with one another: which is impossible (iii. 23). Hence the segment AEB coincides with the segment CFD; and magnitudes which coincide are equal (Ax. 8): therefore it is equal to it. Which was to be proved.

PROP. XXV. PROB.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle. It is required to describe the circle of which it is the segment.

Bisect (i. 10) AC in D; from D draw (i. 11) DB at right angles to AC, cutting the arc of the segment in B; and join AB. Then there will be two cases according as the angles BAD, ABD are, or are not equal.

I. Let the angles BAD, ABD be equal, as in Fig. 1.

Because the angle BAD is equal to the angle ABD, therefore DA is equal (i. 6) to DB; and DA is equal to DC

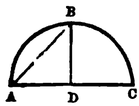


Fig. 1.

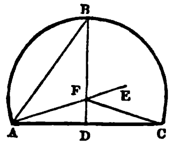


Fig. 2.

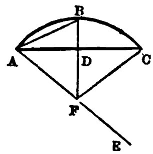


Fig. 3.

by const^n: and things that are equal to the same thing are equal to one another (Ax. 1): hence DB is equal to DC, and the three straight lines DA, DB, DC are equal to one another. And because from the point D more than two equal straight lines DA, DB, DC can be drawn to the arc ABC, B is the centre (iii. 9) of the circle of which ABC is the arc. Hence if with centre D and radius DA, DB or DC a circle be described, it will be the one of which ABC is a segment.

II. Let the angles BAD, ABD be not equal as in Fig^s 2, 3.

At the point A in the straight line AB make (i. 23) the angle BAE equal to the angle ABD: and let F be the point where BD (Fig. 2) or BD produced if necessary (Fig. 3) cuts AE. Join FC.

Then because the angle ABF is equal to the angle BAF, therefore FA is equal to FB. Now AD is equal to DC by const^n, DF common to the two triangles ADF, FDC, and the angles ADF, FDC are equal (Def. 10), because DB is at right angles to AC; therefore these two triangles have the two sides AD, DF respectively equal to the two sides CD, DF, and the included angle ADF equal to the included angle CDF. Therefore they are equal in every respect (i. 4); and hence the base FA is equal to the base FC. But FA is equal to FB; and things that are equal to the same thing are equal to one another: therefore FB is equal to FC, and the three straight lines FA, FB, FC are equal to one another. And because from the point F more than two equal straight lines FA, FB, FC can be drawn to the arc ABC, F is the centre of the circle of which ABC is an arc. Hence if with centre F and radius any one of the three FA, FB, FC a circle be described, it will be the circle of which ABC is a segment.

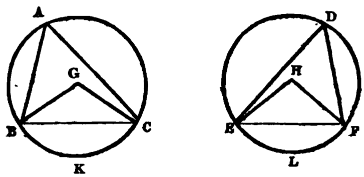
Therefore in every case the circle has been described of which ABC is the segment. Which was to be done.

PROP. XXVI. THEOR.

In equal circles or the same circle, equal angles shall stand on equal arcs, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, of which the centres are G, H respectively; and let BGC, EHF be equal angles at their centres, and BAC, EDF equal angles at their circumferences, standing on the arcs BKC, ELF. Then the arc BKC shall be equal to the arc ELF.

Join BC, EF.

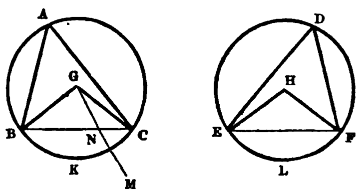


Because the circles ABC, DEF, are equal by hyp^s, their radii are equal (iii. Def. 1). Therefore GB, GC, HE, HF are all equal, and the angles BGC, EHF are equal by hyp^s; therefore the two triangles BGC, EHF have the two sides BG, GC respectively equal to the two sides EH, HF, and the included angle BGC equal to the included angle EHF. Therefore these two triangles are equal in every respect (i. 4); and hence the base BC is equal to the base EF. And because the angle BAC is equal to the angle EDF by hyp^s, the segments BAC, EDF are similar by def^n (iii. Def. 11); and they are on equal straight lines: therefore the segment BAC is equal (iii. 24) to the segment EDF. And by hyp^s the whole circle ABC is equal to the whole circle EDF: therefore taking away equals from equals, the remaining segment BKC is equal (Ax. 3) to the remaining segment ELF, and the arc BKC to the arc ELF. Had the angles been in the same circle ABC, instead of in equal circles, the proof would have been exactly the same, the point H in this case coinciding with G, and D, E, L, F being points in the circumference ABC. Which was to be proved.

PROP. XXVII. THEOR.

In equal circles or the same circle, the angles which stand on equal arcs, whether they be at the centres or circumferences, shall be equal to one another.

Let ABC, DEF be equal circles, of which the centres are G, H respectively; and let the angles BGC, EHF at the centres and the angles BAC, EDF at the circumferences stand on the equal arcs BKC, ELF. Then the angle BGC shall be equal to the angle EHF, and the angle BAC to the angle EDF.



If the angle BGC be equal to the angle EHF: then because the angle BAC at the circumference is half (iii. 20) of the angle BGC at the centre, having the same arc BKC for base, and similarly the angle EDF is half of the angle EHF, and the halves of equal things are equal (Ax. 7): therefore also the angle BAC is equal to the angle EDF. But if the angle BGC be not equal to the angle EHF, one of them must be the greater: let, if possible, BGC be the greater, and at the point G in the straight line BG make (i. 23) the angle BGM equal to the angle EHF. Let GM cut the arc BKC in the point N.

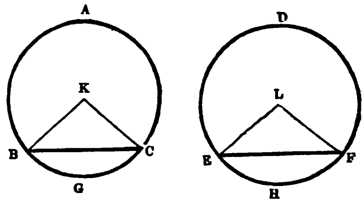
Since the angle BGN is equal to the angle EHF by const^{n}, and in equal circles equal angles at the centre stand on equal arcs (iii. 27): therefore the arc BKN is equal to the arc CLF. But the arc BKC is equal to the arc ELF by hyp^{s}; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the arc BKN is equal to the arc BKC, that is, the part equal to the whole: which is impossible (Ax. 9). Therefore the angle BGC is not unequal to the angle EHF, that is, it is equal to it; and, as was shewn before, the angle BAC is also equal to the angle EDF. Had the arcs been in the same circle ABC, instead of in equal circles, the proof would have been exactly the same, H in this case coinciding with

G, and D, E, L, F being points in the circumference ABC. Which was to be proved.

PROP. XXVIII. THEOR.

In equal circles, and in the same circle, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles; and let BC, EF be equal straight lines in them, which cut off the two greater arcs BAC, EDF and the two less arcs BGC, EHF. Then the greater arc BAC shall be equal to the greater EDF, and the less BGC to the less EHF.



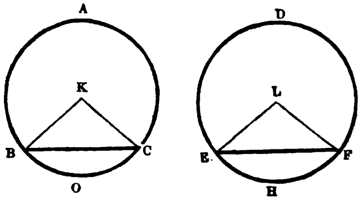
Find (iii. 1) the centre K of the circle ABC, and the centre L of the circle EDF. Join KB, KC, LE, LF.

Because the circles ABC, DEF, are equal by hyp[s], their radii are equal (iii. Def. 1). Therefore KB, KC, LE, LF are all equal, and BC is equal to EF by hyp[s]; therefore the two triangles KBC, LEF have the three sides BK, KC, CB respectively equal to the three sides EL, LF, FE. Therefore these two triangles are equal in every respect (i. 8); and hence the angle BKC is equal to the angle ELF. But in equal circles, equal angles at the centre stand on equal arcs (iii. 26): therefore the arc BGC is equal to the arc EHF. But the whole circumference ABC is equal to the whole circumference DEF, since the circles are equal (iii. Def. 1): therefore, taking away equals from equals, the remaining arc BAC is equal (Ax. 3) to the remaining arc EDF. Hence the greater arc BAC is equal to the greater EDF, and the less BGC to the less EHF. Had the straight lines been in the same circle ABC instead of in equal circles, the proof would have been exactly the same, L in this case coinciding with K, and D, E, H, F being points in the circumference ABC. Which was to be proved.

PROP. XXIX. THEOR.

In equal circles, and in the same circle, equal arcs shall be subtended by equal straight lines.

Let ABC, DEF be equal circles, and let BGC, EHF be equal arcs in them, subtended by the straight lines BC, EF. Then BC shall be equal to EF.



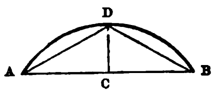
Find (iii. 1) the centre K of the circle ABC, and the centre L of the circle DEF. Join KB, KC, LE, LF.

The arc BGC is equal to the arc EHF by hyp^s; and in equal circles, the angles at the centre which stand on equal arcs are equal (iii. 27): therefore the angle BKC is equal to the angle ELF. Also because the circles ABC, DEF are equal by hyp^s, their radii are equal (iii. Def. 1), and hence BK, KC, EL, LF are all equal; therefore the two triangles BKC, ELF have the two sides BK, KC respectively equal to the two sides EL, LF, and the included angle BKC equal to the included angle ELF. Therefore these two triangles are equal in every respect (i. 4); and hence the base BC is equal to the base EF. Had the arcs been in the same circle ABC, instead of in equal circles, the proof would have been exactly the same, L in this case coinciding with K, and D, E, H, F being points in the circumference ABC. Which was to be proved.

PROP. XXX. THEOR.

To bisect a given arc of a circle, i. e. to divide it into two equal arcs.

Let ADB be the given arc of a circle. It is required to bisect it.



Join the extremities of the arc A, B. Bisect (i. 10) AB in C, and from C draw (i. 11) CD at right

angles to AB, cutting the arc in D. Then the arc ADB shall be bisected in the point D.

Join DA, DB.

Because AC is equal to CB by const^n, CD common to the two triangles ACD, BCD, and the angles ACD, BCD, equal, since CD is at right angles to AB (Def. 10): therefore these two triangles have the two sides AC, CD respectively equal to the two sides BC, CD, and the included angle ACD equal to the included angle BCD. Therefore they are equal in every respect (i. 4); and hence the base AD is equal to the base BD. Now in the same circle equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less (iii. 28); and the arcs AD, BD cut off by the equal straight lines AD, BD, are each of them less than a semicircle, because the straight line CD bisects the straight line in the circle AB at right angles, and therefore, if produced, passes through (iii. 1. Cor.) the centre, and is a diameter of the circle of which ADB is an arc: hence the arc AD is equal to the arc DB. Therefore the given arc ADB is bisected in the point D. Which was to be done.

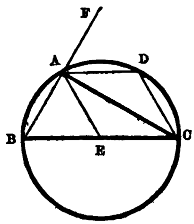
PROP. XXXI. THEOR.

In a circle, the angle in a semicircle shall be a right angle; the angle in a segment greater than a semicircle shall be less than a right angle; and the angle in a segment less than a semicircle shall be greater than a right angle.

Let ABC be a circle, and from one of the extremities C of any diameter BC let CA be drawn cutting the circle in A, so that BAC is a semicircle, the segment ABC greater than a semicircle, and the segment ADC less than a semicircle. Then the angle in the semicircle BAC shall be a right angle; the angle in the segment ABC less than a right angle; and the angle in the segment ADC greater than a right angle.

Bisect (i. 10) BC in E; then E is the centre of the circle. Join EA, BA; and produce BA to F. Also in the arc ADC take any point D, and join DA, DC.

I. The angle BAC in the semicircle shall be a right angle.



By the def^n of a circle BE, AE, CE are all equal. Now because EB is equal to EA, the angle EBA is equal (i. 5) to the angle EAB; and because EA is equal to EC, the angle ECA is equal to the angle EAC: hence, adding equals to equals, the whole angle BAC is equal (Ax. 2) to the two angles ABC, ACB. But because the side BA of the triangle ABC is produced to F, the exterior angle FAC is equal (i. 32) to the two interior and opposite angles ABC, ACB; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle BAC is equal to the angle FAC. That is, the straight line CA standing on the straight line BF makes with it the adjacent angles CAF, CAB equal to one another: therefore by def^n (Def. 10) each of these angles is a right angle. Hence the angle BAC is a right angle.

II. The angle ABC in the segment ABC shall be less than a right angle.

The two angles ABC, BAC of the triangle ABC are less than two right angles (i. 17); and BAC has been shewn to be a right angle: therefore the remaining angle ABC is less than a right angle.

III. The angle ADC in the segment ADC shall be greater than a right angle.

Because ABCD is a quadrilateral figure inscribed in a circle, each pair of its opposite angles are equal to two right angles (iii. 22). Hence the angles ABC, ADC are equal to two right angles; and ABC has been shewn to be less than a right angle: therefore the remaining angle ADC is greater than a right angle.

Hence the angle in the semicircle BAC is a right angle; the angle in the segment ABC greater than a semicircle is less than a right angle; and the angle ADC in the segment less than a semicircle is greater than a right angle. Which was to be proved.

COR.—If one angle of a triangle be equal to the other two: then it shall be a right angle.

For let ABC be a triangle, one of the angles of which BAC is equal to the other two ABC, ACB. Then it may be shewn by producing BA to F as in the first part of the propⁿ that BAC is a right angle. Which was to be proved.

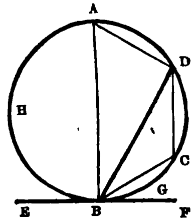
PROP. XXXII. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle: then the angles which this line makes with the touching line shall be respectively equal to the angles in the alternate segments of the circle.

Let EF touch the circle AGH, and from the point of contact B let BD be drawn, cutting the circle in D and dividing it into the segments DGB, DHB. Then the angles DBF, DBE, which BD makes with EF, shall be respectively equal to the angles in the alternate segments; that is, the angle DBF shall be equal to the angle in the segment DHB, and the angle DBE to the angle in the segment DGB.

From B draw (i. 11) BA at right angles to EF, cutting the circle in A; in the arc BGD take any point C; and join AD, DC, CB.

I. The angle FBD shall be equal to the angle in the segment DHB.



Because EF touches the circle, and from the point of contact B, BA is drawn at right angles to EF, the centre of the circle is in BA (iii. 19), and therefore AGB is a semicircle. Hence the angle ADB in the semicircle is a right angle (iii. 31). Now the three angles BAD, ADB, DBA of the triangle ABD are equal (i. 32) to two right angles; and one of them ADB is a right angle: therefore the other two ABD, BAD are equal to a right angle. But ABF is a right angle by constⁿ; therefore the angles ABD, BAD are

equal to the angle ABF. From each of these equals take away the common angle ABD: then the remaining angle DBF is equal (Ax. 3) to the remaining angle BAD.

II. The angle DBE shall be equal to the angle in the segment BGD.

Because ABCD is a quadrilateral figure inscribed in a circle, each pair of its opposite angles are equal to two right angles (iii. 22): therefore the angles BAD, BCD are equal to two right angles. Also because DB makes with EF on the same side of it the adjacent angles DBE, DBF, these angles are equal (i. 13) to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles DBE, DBF are equal to the angles BCD, BAD. But the angle DBF has been proved equal to the angle BAD; hence, taking away equals from equals, the remaining angle DBE is equal to the remaining angle BCD.

Hence the angle DBF is equal to the angle in the alternate segment DHB, and the angle DBE to the angle in the alternate segment DGB. Which was to be proved.

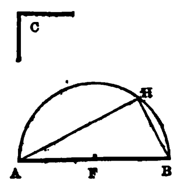
PROP. XXXIII. PROB.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.

Let AB be the given straight line, and C the given angle. It is required on AB to describe a segment of a circle, which shall contain an angle equal to the angle C.

There are two cases, according as the angle C is a right angle, or is not.

I. Let the angle C be a right angle.



Bisect (i. 10) AB in F; and with centre F and radius FA or FB, describe the semicircle AHB. Then if a point H be taken in the semicircle, and HA, HB be joined, the angle in the semicircle AHB is a right angle, and therefore equal to the angle C: that is, the angle in the segment AHB is equal to the angle C.

II. Let the angle C be not a right angle.

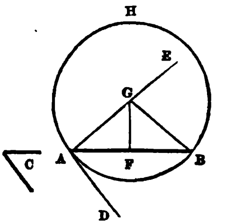


Fig. 1.

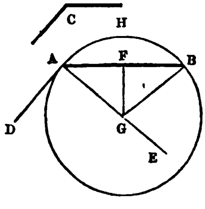


Fig. 2.

At the point A in the straight line AB make (i. 23) the angle BAD equal to the angle C, and from A draw (i. 11) AE at right angles to AD, AB falling between AE and AD if BAD is less (Fig. 1) or AE between AB and AD, if it is greater than a right angle (Fig. 2). Bisect AB in F, and from F draw FG at right angles to AB, cutting AE in G; join GB. Then because AF is equal to FB by const^{n}, FG common to the two triangles GAF, GBF, and the angles GFA, GFB equal, since FG is at right angles to AB (Def. 10): therefore these two triangles have the two sides AF, FG respectively equal to the two sides BF, FG, and the included angle AFG equal to the included angle BFG. Therefore they are equal in every respect (i. 4); and hence the base GA is equal to the base GB. Therefore the angle described with centre G and either GA or GB as a radius, will pass through the extremity of the other: let this circle be the circle AHB. Then the segment AHB on the opposite side of AB to AD shall be the segment required.

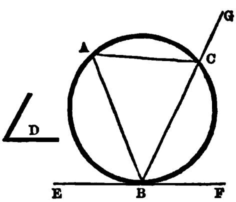
Because AD is drawn perpendicular to the radius GA of the circle AHB from its extremity A, therefore AD touches (iii. 16, Cor.) the circle at A; and because AB is drawn from the point of contact A cutting the circle in B, the angle DAB is equal to the angle in the alternate segment AHB. But the angle DAB is equal to the angle C by const^{n}; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle in the segment AHB is equal to the angle C.

Hence in both cases, on the given straight line AB a segment AHB of a circle is described containing an angle equal to the given angle C. Which was to be done.

PROP. XXXIV. PROB.

From a given circle to cut off a segment which shall contain an angle equal to a given angle.

Let ABC be the given circle, and D the given angle. It is required from the circle ABC to cut off a segment which shall contain an angle equal to the angle D.



In the circumference of the circle take any point B, and through B draw (iii. 17) EF touching the circle; at B, in the straight line BF, make (i. 23) the angle FBG equal to the angle D, and let BG cut the circle in C, dividing it into two segments. Then the segment BAC on the side of BC, opposite to BF, shall be the one required.

Because EF touches the circle, and from the point of contact B, BC is drawn cutting the circle in C, the angle FBC is equal (iii. 32) to the angle in the alternate segment BAC. But the angle FBC is equal to the angle D by const^{n}; and things that are equal to the same thing are equal to one another (Ax. 1); therefore the angle in the segment BAC is equal to the angle D. Hence from the given circle ABC a segment BAC has been cut off containing an angle equal to the given angle D. Which was to be done.

PROP. XXXV. THEOR.

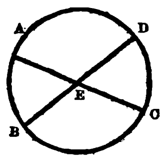
If through any point within a circle two straight lines be drawn in the circle: then the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Let E be any point within the circle ABC, and through E let the straight lines AC, BD be drawn in the circle. Then the rectangle contained by AE, EC, the segments of AC, shall be equal to the rectangle contained by BE, ED, the segments of BD.

There are four cases, according as (1) both the straight

lines pass through the centre; (2) one of them passes through the centre and cuts the other, which does not pass through the centre, at right angles; (3) one of them passes through the centre and cuts the other which does not pass through the centre, but not at right angles; (4) or neither of the straight lines passes through the centre.

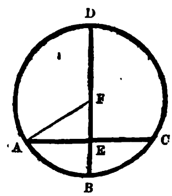
I. Let both AC and BD pass through the centre so that E is the centre.



Then by the def^n of a circle, EA, EB, EC, ED are all equal: therefore the rectangle AE, EC is equal to the rectangle BE, ED.

II. Let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre at right angles, in E.

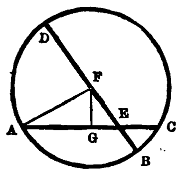
Bisect (i. 10) BD in F, and join AF: then F is the centre of the circle.



Because BD which passes through the centre, cuts the straight line AC in the circle at right angles, it also bisects (iii. 3) it, and AE is equal to EC. And because BD is bisected in F, and divided into two unequal parts in E, the rectangle BE, ED together with the square of EF is equal (ii. 5) to the square of FB, that is, to the square of FA, because FB is equal to FA by the def^n of a circle. But the squares of FE, EA are equal (i. 47) to the square of FA, because FEA is a right angle; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the rectangle BE, ED together with the square of EF is equal to the squares of AE, EF. From each of these equals take away the common square of EF: then the remaining rectangle BE, ED is equal (Ax. 3) to the square of AE. But the square of AE is the rectangle AE, EC, because AE is equal to EC: therefore the rectangle BE, ED is equal to the rectangle AE, EC.

III. Let one of them BD pass through the centre, and cut the other AC which does not pass through the centre, in E, but not at right angles.

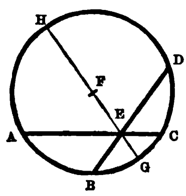
Bisect BD in F: then F is the centre of the circle. Join AF, and from F draw (i. 12) FG perpendicular to AC.



Because FG, which passes through the centre, cuts the straight line in the circle AC, which does not pass through the centre, at right angles, it also bisects it, and AG is equal to GC. And because AC is bisected in G, and divided into two unequal parts in E, the rectangle AE, EC together with the square of GE is equal to the square of AG. To each of these equals add the square of GF; then the rectangle AE, EC together with the squares of EG, GF is equal (Ax. 2) to the squares of AG, GF. But the squares of EG, GF are equal to the square of EF, because FGE is a right angle; and the squares of AG, GF are equal to the square of AF, because AGF is a right angle: therefore the rectangle AE, EC together with the square of EF, is equal to the square of AF, that is, to the square of FB, since AF is equal to FB by the def^{n} of a circle. Again because DB is bisected in F, and divided into two unequal parts in E, the rectangle BE, ED together with the square of EF is equal to the square of FB; and things that are equal to the same thing are equal to one another: therefore the rectangle BE, ED together with the square of FE is equal to the rectangle AE, EC together with the square of EF. From each of these equals take away the common square of EF: then the remaining rectangle BE, ED is equal to the remaining rectangle AE, EC.

IV. Let neither of the straight lines AC, BD pass through the centre.

Find (iii. 1) the centre F of the circle; join EF; and produce EF both ways to meet the circle in G and H.



Because the two straight lines in the circle AC, GH cut one another in E, of which one GH passes through

the centre, therefore by the third case the rectangle AE, EC is equal to the rectangle GE, EH. For like reason, the rectangle BE, ED is also equal to the rectangle GE, EH; and things that are equal to the same thing are equal to one another: therefore the rectangle AE, EC is equal to the rectangle BE, ED.

Hence it has been shewn in every case that the rectangle AE, EC is equal to the rectangle BE, ED. Which was to be proved.

PROP. XXXVI. THEOR.

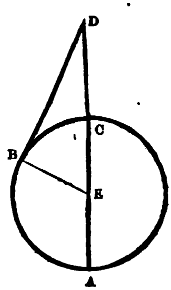
If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it: then the rectangle contained by the whole line cutting the circle and the part of it without the circle shall be equal to the square of the touching line.

Let D be any point without the circle ABC, and from D let the straight lines DCA, DB be drawn, one of which DCA cuts the circle in C and A, and the other DB touches the circle at B. Then the rectangle contained by the whole of the cutting line AD and the part of it DC without the circle shall be equal to the square of the touching line DB.

There are two cases according as the cutting line DCA passes through the centre or not.

I. Let DCA pass through the centre.

Bisect (i. 10) CA in E; then E is the centre of the circle. Join BE.

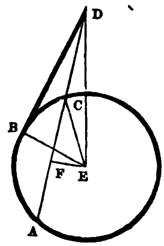


Because DB touches the circle by hyp[s], and EB is drawn from the centre E to the point of contact B, the angle DBE is a right angle (iii. 18). And because AC is bisected in E, and produced to D, the rectangle AD, DC together with the square of CE is equal (ii. 6) to the square of ED. But the square of EC is equal to the square of EB, since EC is equal to EB by the def[n] of a circle; and the square of DE is equal (i. 47) to the squares of DB, BE, because

DBE is a right angle: therefore the rectangle AD, DC together with the square of EB is equal to the squares of DB, BE. From each of these equals take away the common square of BE: then the remaining rectangle AD, DC is equal (Ax. 3) to the remaining square of DB.

II. Let DCA not pass through the centre.

Find (iii. 1) the centre E: from E draw (i. 12) EF perpendicular to CA; and join EB, EC, ED.

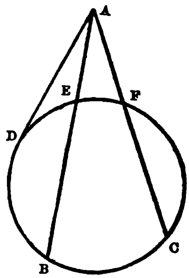


For the same reason as before the angle DBE is a right angle; and because EF which passes through the centre cuts the straight line in the circle CA, which does not pass through the centre at right angles, it also bisects (iii. 3) it, and hence CF is equal to FA. Now because CA is bisected in F and produced to D, the rectangle AD, DC together with the square of CF is equal to the square of DF. To each of these equals add the square of FE; then the rectangle AD, DC together with the squares of CF, FE is equal (Ax. 2) to the squares of DF, FE. But the squares of CF, FE are equal to the square of CE because CFE is a right angle, and the squares of DF, FE are equal to the square of DE, because DFE is a right angle: therefore the rectangle AD, DC together with the square of EC is equal to the square of DE. But the square of EC is equal to the square of EB, since EC is equal to EB by the defn of a circle; and the square of DE is equal to the squares of DB, BE because DBE is a right angle: therefore the rectangle AD, DC together with the square of EB is equal to the squares of DB, BE. From each of these equals take away the common square of EB: then the remaining rectangle AD, DC is equal to the remaining square of DB.

Hence it has been shewn in both cases that the rectangle AD, DC is equal to the square of DB. Which was to be proved.

COR.—If from any point without a circle, there be drawn two straight lines cutting it: then the rectangles contained by the whole lines and the parts

of them without the circle shall be equal to one another.



Let A be any point without the circle BEF; and from A let AEB, AFC be drawn cutting the circle in E, B and F, C respectively. Then the rectangle BA, AE shall be equal to the rectangle CA, AF.

From A draw (iii. 17) AD touching the circle at D.

Then by the prop^n each of the rectangles BA, AE and CF, FA is equal to the square of the touching line AD; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the rectangle BA, AE is equal to the rectangle CA, AF. Which was to be proved.

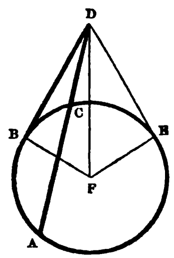
PROP. XXXVII. THEOR.

If from any point without a circle two straight lines be drawn, one of which cuts the circle and the other meets it, such that the rectangle contained by the whole line cutting the circle and the part of it without the circle is equal to the square of the meeting line: then the meeting line shall touch the circle.

Let D be any point without the circle ABC, and from D let the straight lines DCA, DB be drawn, one of which DCA cuts the circle in C and A, and the other DB meets the circle in B, so that the rectangle contained by the whole of the cutting line DA and the part of it DC without the circle is equal to the square of the meeting line DB. Then DB shall touch the circle at B.

Find (iii. 1) the centre F of the circle ABC; join DF; and from D draw (iii. 17) the straight line DE touching the circle in E on the side of DF opposite to DB. Join FB, FE.

Because DE touches the circle by const^n and FE is drawn from the centre F to the point of contact E, the angle



FED is a right angle (iii. 18). And because from the point D without the circle, the two straight lines DCA, DE are drawn, one of which DCA cuts the circle in C and A, and the other DE touches it in E, therefore the rectangle AD, DC is equal (iii. 36) to the square of DE. But the rectangle AD, DC is equal to the square of DB by hyp^s; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the square of DE is equal to the square of DB, and hence DE is equal to DB. Also FB is equal to FE by the def^n of a circle, and FD is common to the two triangles DEF, DBF: therefore these two triangles have the three sides DE, EF, FD respectively equal to the three sides DB, BF, FD. Therefore they are equal in every respect (i. 8); and hence the angle DEF is equal to the angle DBF. But DEF is a right angle: therefore also DBF is a right angle. Hence BD is drawn perpendicular to the radius BF of the circle from its extremity B: therefore BD touches (iii. 16, Cor.) the circle ABC at B. Which was to be proved.

THE

ELEMENTS OF EUCLID.

BOOK IV.

DEFINITIONS.

I.

A POLYGON is said to be inscribed in another polygon of the same number of sides, when each of the angular points of the former is in a side of the latter.

II.

A polygon is said to be circumscribed about another polygon of the same number of sides, when each of the sides of the former passes through an angular point of the latter.

III.

A polygon is said to be inscribed in a circle, when all its angular points are in the circumference of the circle.

IV.

A polygon is said to be circumscribed about a circle, when each of its sides touches the circle.

V.

A circle is said to be inscribed in a polygon, when it is touched by each of the sides of the polygon.

VI.

A circle is said to be circumscribed about a polygon, when the circumference of the circle passes through all the angular points of the polygon.

VII.

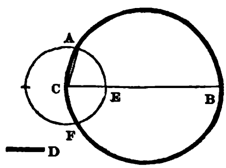
A straight line is said to be placed in a circle, when its extremities are in the circumference of the circle.

PROPOSITIONS.

PROP. I. PROB.

In a given circle to place a straight line, equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line which is not greater than the diameter of the circle. It is required to place in the circle ABC a straight line equal to D.



Find (iii. 1) the centre of the circle ABC, and through it draw any diameter BC: then if BC be equal to D, the thing required is already done. But if not: BC must be greater than D by hyp[s]; from CB cut off (i. 3) CE equal to D, and with centre C and radius CE describe the circle FEA, cutting the circle ABC in A. Join CA. Then CA shall be the straight line required.

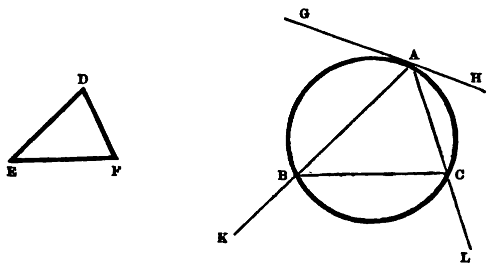
By the def[n] of a circle CA is equal to CE; and D is equal to CE by const[n]; and things that are equal to the same thing are equal to one another (Ax. 1): therefore CA is equal to D, and its extremities are points in the circumference of the circle ABC. Hence in the given circle ABC has been placed (iv. Def. 7) a straight line AC equal to the given straight line D. Which was to be done.

PROP. II. PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given tri-

angle. It is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.



In the circumference of the circle ABC take any point A, and through A draw (iii. 17) the straight line GAH touching the circle. At the point A in the straight line AH make (i. 23) the angle HAL equal to the angle DEF, and at the point A in the straight line AG make the angle GAK equal to the angle DFE. Let AK, AL cut the circle in B, C respectively; and join BC. Then ABC shall be the triangle required.

Because GH touches the circle ABC by constⁿ, and from the point of contact A, AC is drawn cutting the circle in C, the angle HAC is equal (iii. 32) to the angle ABC in the alternate segment ABC. But the angle HAC is equal to the angle DEF by constⁿ; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle ABC is equal to the angle DEF. By like reasoning it may be shewn that the angle ACB is equal to the angle DFE: therefore the remaining angle BAC is equal (i. 32, Cor. A) to the remaining angle EDF. Hence the triangle ABC is equiangular to the given triangle DEF, and it is inscribed (Def. 3) in the given circle ABC, since all its angular points are in the circumference of the circle. Which was to be done.

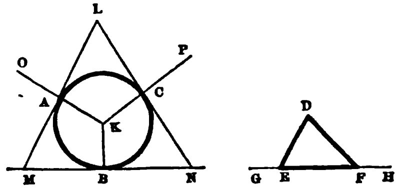
PROP. III. PROB.

About a given circle to circumscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given tri-

angle. It is required about the circle ABC to circumscribe a triangle equiangular to the triangle DEF.

Produce EF both ways to the points G, H: find (iii. 1) the centre K of the circle ABC, and draw any radius KB. At the point K in the straight line KB make



(i. 23) the angle BKO equal to the angle DEG, and the angle BKP equal to the angle DFH. Let KO, KP cut the circle in A, C; and through A, B, C draw (iii. 17) LM, MN, NL touching the circle ABC and forming a triangle LMN. Then LMN shall be the triangle required.

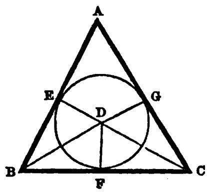
Because LM, MN, NL touch the circle by const^n, and KA, KB, KC are drawn from the centre K to the points of contact A, B, C; therefore the angles at the points A, B, C are right angles (iii. 18). Also the four angles of the quadrilateral figure KAMB together with four right angles are equal (i. 32, Cor. 1) to twice as many right angles as the figure has sides, that is, to eight right angles: from each of these equals take away four right angles: therefore the remaining four angles of KAMB are equal to (Ax. 3) the remaining four right angles. But of these two, viz. KAM, KBM are right angles; therefore the other two, viz. AKB, AMB are equal to two right angles. And because DE makes with GF on the same side of it the adjacent angles DEG, DEF, these angles are equal (i. 13) to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles AKB, AMB are equal to the angles DEG, DEF. And by const^n the angle AKB is equal to the angle DEG: hence, taking away equals from equals, the remaining angle AMB is equal to the remaining angle DEF. In like manner, the angle CNB may be shewn to be equal to the angle DFE; and the third angle MLN is equal (i. 32, Cor. A) to the third angle EDF. Hence the triangle LMN is equiangular to the given triangle DEF, and it is circumscribed (iv.

Def. 4) about the given circle ABC, since each of its sides touches the circle. Which was to be done.

PROP. IV. THEOR.

To inscribe a circle in a given triangle.

Let ABC be the given triangle. It is required to inscribe a circle in the triangle ABC.



Bisect (i. 9) the angles ABC, BCA by the straight lines BD, CD, meeting one another in D; and from D draw (i. 12) DE, DF, DG perpendicular to AB, BC, CA.

Because the angle EBD is equal to the angle FBD by const^n; the right angle DEB to the right angle DFB, since all right angles are equal (Ax. 11); and DB common to the two triangles DEB, DFB: therefore these two triangles have the two angles DEB, EBD respectively equal to the two angles DFB, FBD, and the side DB opposite to the équal angles DEB, DFB common to each. Therefore they are equal in every respect (i. 26); and hence the side DE is equal to the side DF. For like reason, DG is equal to DF; and things that are equal to the same thing are equal to one another (Ax. 1): therefore DE is equal to DG, and the three straight lines DE, DF, DG are all equal. Hence the circle described with centre D, and any one of them as radius will pass through the extremities of the other two; and since the angles at E, F, G are right angles by const^n, the straight lines AB, BC, CA are drawn perpendicular to the radii DE, DF, DG of the circle from their extremities E, F, G, and therefore touch (iii. 16, Cor.) the circle EFG. Therefore in the given triangle ABC has been inscribed (iv. Def. 5) the circle EFG. Which was to be done.

PROP. V. PROB.

To circumscribe a circle about a given triangle.

Let ABC be the given triangle. It is required to circumscribe a circle about the triangle ABC.

Bisect (i. 10) AB, AC in D and E; and from D, E draw (i. 11) DG, EH at right angles to AB, AC. Then, joining ED, since the angles ADG, AEH are two right angles, the angles EDG, DEH are less (Ax. 9) than two right angles. That is, the straight line DE cutting the two straight lines DG, EH in D, E, makes the two interior angles EDG, DEH, on the same side of DE, together less than two right angles: therefore by the axiom (Ax. 12), DG, EH, being continually produced, shall at length meet in some point on the side of DE towards G, H. Let them be produced to meet in F; join FA; then if F be not (as in Fig. 2) a point in BC, but (as in Figs 1, 3) falls without BC, join also FB, FC.

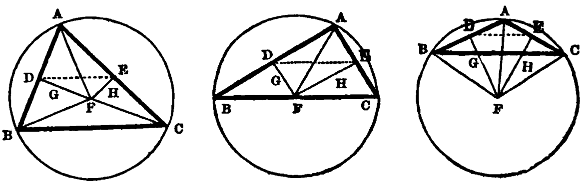


Fig. 1. Fig. 2. Fig. 3.

Because AD is equal to DB by const^n, DF common to the two triangles ADF, BDF, and the angle FDA equal (Def. 10) to the angle FDB, since FD is at right angles to AB: therefore the two triangles ADF, BDF have the two sides AD, DF respectively equal to the two sides BD, DF, and the included angle ADF equal to the included angle BDF. Therefore they are equal in every respect (i. 4); and hence the base AF is equal to the base BF. In like manner it may be shewn that CF is equal to FA; and things that are equal to the same thing are equal to one another (Ax. 1): therefore BF is equal to CF, and the three straight lines AF, BF, CF are all equal. Hence the circle described with the centre F and any one of them as radius, will pass through the extremities of the other two; and as the circle ABC passes through the angular points of the given triangle ABC, it is circumscribed (iv. Def. 6) about it. Which was to be done.

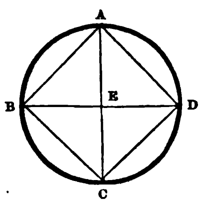
COR.—When the centre F of the circle falls within the triangle ABC (Fig. 1), each of its angles is an angle in a segment greater than a semicircle, and is therefore an acute angle (iii. 31); when F falls on a side BC of the triangle (Fig. 2), BC is a diameter, and the angle BAC is an angle in a semicircle, and therefore a right angle; and when F falls without the triangle ABC (Fig. 3), the side BC beyond which it is, cuts off a segment less than a semicircle, and therefore the angle BAC in it is greater than a right angle. Hence:—

If the given triangle be acute-angled, the centre of the circle falls within the triangle; if right-angled, the centre is in the side opposite to the right angle; and if obtuse-angled, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROP. VI. PROB.

To inscribe a square in a given circle.

Let ABC be the given circle. It is required to inscribe a square in the circle ABC.



Find (iii. 1) the centre E of the circle ABC; and through E draw two diameters AC, BD at right angles (i. 11) to one another. Join AB, BC, CD, DA. Then the quadrilateral figure ABCD shall be the square required.

By the def^{n} of a circle, EA, EB, EC, ED are all equal. Because BE is equal to ED, AE common to the two triangles AEB, AED, and the right angle AEB equal to the right angle AED, since all right angles are equal (Ax. 11); therefore these two triangles have the two sides BE, EA respectively equal to the two sides DE, EA, and the included angle BEA equal to the included angle DEA. Therefore they are equal in every respect (i. 4); and hence the base BA is equal to the base AD. In like manner it may be shewn that BA is equal to BC, and AD to DC: therefore ABCD is equilateral.

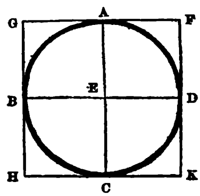
Again because BD is a diameter, the angle BAD is an angle in a semicircle, and therefore a right angle (iii. 31); for the same reason each of the angles ABC, BCD, CDA is a right angle: therefore ABCD is rectangular. Hence the quadrilateral figure ABCD is both equilateral and rectangular, and is therefore a square (Def. 30); and since all its angular points ABCD are in the circumference, it is inscribed (iv. Def. 3) in the given circle ABC. Which was to be done.

PROP. VII. PROB.

To circumscribe a square about a given circle.

Let ABC be the given circle. It is required to circumscribe a square about the circle ABC.

Find (iii. 1) the centre E of the circle ABC; and through E draw (i. 11) two diameters AC, BD at right angles to one another. Through A, B, C, D draw FG, GH, HK, KF touching the circle, and forming the quadrilateral figure FGHK. Then FGHK shall be the square required.



Because FG touches the circle ABC, and EA is drawn from the centre E to the point of contact A, the angles at A are right angles (iii. 18): for like reason the angles at B, C, D are right angles. Now because AEB is a right angle, as likewise is EBG, EB cutting the two straight lines GH, AC in B and E makes the two interior angles on the same side of BE, GBE, AEB equal to two right angles: therefore GH is parallel (i. 28) to AC, and for like reason FK is parallel to AC. Now straight lines which are each of them parallel to the same straight line are parallel (i. 30); therefore GH, FK are parallel. In the same manner it may be shewn that GF, HK are each of them parallel to BD, and therefore to one another: hence each of the figures GK, GC, CF, FB, BK is a parallelogram. And because the opposite sides of parallelograms are equal (i. 34), GF is equal to HK, and GH to FK, and also AC is equal to each of the two GH, FK, and BD to each of the two GF, HK.

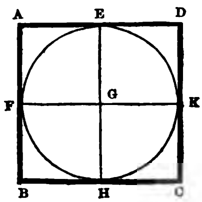
But the diameter AC is equal to the diameter BD: therefore GH, FK are each of them equal to GF or HK, and the four straight lines GF, FK, KH, HG are all equal, or FGHK is equilateral. Again, because the opposite angles of parallelograms are equal, the angle AGB of the parallelogram AGBE is equal to the angle AEB; but AEB is a right angle; therefore likewise the angle at G is a right angle. And since the angle at G, one of the angles of the parallelogram GK, is a right angle, all its angles are right angles, (I. 46, Cor.) or FGHK is rectangular. Hence the quadrilateral figure FGHK is both equilateral and rectangular, and is therefore a square (Def. 30); and since each of its sides touches the given circle ABC, it is circumscribed (IV. Def. 4) about it. Which was to be done.

PROP. VIII. PROB.

To inscribe a circle in a given square.

Let ABCD be the given square. It is required to inscribe a circle in the square ABCD.

Bisect (I. 10) two of the adjacent sides AD, AB of the square in E and F; through E draw (I. 31) EH parallel to AB or CD, cutting BC in H; and through F draw FGK parallel to AD or BC, cutting EH in G, and CD in K.



Each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram by constn, and as each contains an angle of the square ABCD, and therefore (Def. 30) a right angle, each is rectangular (I. 46, Cor.); and by constn AE is the half of AC, and AF the half of AB. But AD is equal to AB, since they are sides of the square ABCD; and the halves of equal things are equal (Ax. 7): therefore AE is equal to AF. And because the opposite sides of parallelograms are equal (I. 34), AE is equal to FG, and AF to EG: therefore FG is equal to GE. In the same manner it may be shewn that GE is equal to GK, GK to GH, GH to GF: therefore the four straight lines GE, GF, GH, GK are equal. Hence the circle described with the

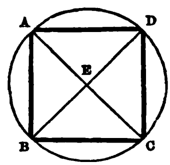
centre G and any one of them as radius, will pass through the extremities of the other three; and the straight lines AB, BC, CD, DA will touch this circle EFH at E, F, H, K, because they are each drawn perpendicular to a radius from its extremity (iii. 16, Cor.), the angles at these points being angles of a rectangle, and therefore right angles. Therefore in the given square ABCD has been inscribed (iv. Def. 5) the circle EFH. Which was to be done.

PROP. IX. PROB.

To circumscribe a circle about a given square.

Let ABCD be the given square. It is required to circumscribe a circle about the square ABCD.

Draw the diagonals AC, BD of the square, cutting one another in E.



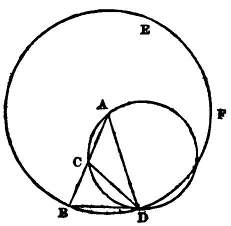
Because DA is equal to AB, since they are sides of the square ABCD (Def. 30), and BC equal to CD for the same reason; and AC is common to the two triangles ABC, ACD: therefore these two triangles have the three sides AB, BC, CA respectively equal to the three sides AD, DC, CA. Therefore they are equal in every respect (i. 8); and hence the angle BAC is equal to the angle CAD, that is, the angle BAD is bisected by AC. In like manner it may be shewn that the angles ABC, BCD, DCA are respectively bisected by BD, CA, DB. Now because the angles DAB, ABC are right angles, since they are angles of the square ABCD, and all right angles are equal to one another (Ax. 11); therefore the angle DAB is equal to the angle ABC. And the halves of equal things are equal (Ax. 7): therefore the angle EAB is equal to the angle EBA, and therefore EA is equal (i. 6) to EB. In like manner it may be shewn that EA is equal to EB, and EC equal to each of EB, ED: therefore the four straight lines EA, EB, EC, ED are all equal. Hence the circle described with centre E and radius any one of them will pass through the extremities of the other three; and since

this circle ABC passes through all the angular points of the given square ABCD, it is circumscribed (iv. Def. 6) about it. Which was to be done.

PROP. X. PROB.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and divide (ii. 11) it in the point C so that the rectangle AB, BC may be equal to the square of AC, the part of AB adjacent to its extremity A. With this extremity A as centre, and AB as radius, describe the circle BEF; in the circle BEF place (iv. 1) the straight line BD equal to AC, which being a part of the radius is therefore less than the diameter of the circle, so that one of its extremities coincide with B. Join AD. Then the triangle ABD shall be such as is required; that is, it shall be isosceles, and each of the angles ABD, ADB at the base BD shall be double of the third angle BAD.



Join CD; and about the triangle ACD circumscribe (iv. 5) the circle ACD.

By const^n the rectangle AB, BC is equal to the square of AC, that is, to the square of BD, because BD is equal to CA by const^n. Hence from the point B without the circle ACD two straight lines BCA, BD are drawn, one of which BCA cuts the circle in C and A, and the other BD meets it at D, such that the rectangle AB, BC is equal to the square of BD: therefore BD touches (iii. 37) the circle ACD at D. Again because BD touches the circle ACD, and from the point of contact D, DC is drawn cutting the circle in C, the angle BDC is equal (iii. 32) to the angle DAC in the alternate segment DAC of the circle. To each of these equals add the angle CDA; therefore the whole angle BDA is equal (Ax. 2) to the angles DAC, CDA. But because the side AC of the triangle ADC is produced to B, the exterior angle BCD is equal (i. 32) to the two interior and opposite angles DAC, CDA; and things that are equal to the same

thing are equal to one another (Ax. 1): therefore the angle BDA is equal to the angle BCD. Also by the def^n of a circle AB is equal to AD; hence the triangle ABD is isosceles (Def. 25), and the angles ABD, ADB at its base are equal (i. 5). Therefore each of the angles ABD, ADB is equal to the angle BCD:

Again because the angle DCB is equal to the angle DBC, CD is equal (i. 6) to BD; but AC is equal to BD; and things that are equal to the same thing are equal to one another: therefore CD is equal to CA, and therefore the angle CDA is equal to the angle CAD. But it was shewn that the angle BCD is equal to the angles CAD, CDA; therefore the angle BCD is double of the angle CAD:

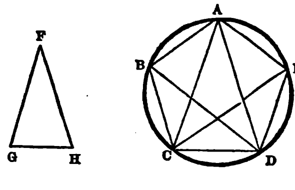
Now it has been proved that each of the angles ABD, ADB is equal to the angle BCD; hence each of the angles ABD, ACB at the base BD of the isosceles triangle ABD is double of the third angle BAD. Therefore the triangle ABD is such as was required. Which was to be done.

PROP. XI. PROB.

To inscribe a regular (i. e. an equilateral and equiangular) pentagon in a given circle.

Let ABC be the given circle. It is required to inscribe a regular pentagon in the circle ABC.

Describe (iv. 10) an isosceles triangle FGH, having each of the angles G, H at its base double of the third angle F. In the circle ABC inscribe (iv. 2) the triangle ACD equiangular to the triangle FGH, the angle CAD being equal to the angle F, and each of the angles ACD, ADC to the angle G or H. Bisect (i. 9) the angles ACD, CDA by the straight lines CE, DB, cutting the circumference of the circle in B and E. Join AB, BC, DE, EA. Then the five-sided figure ABCDE shall be the pentagon required.



Because the triangle ACD is equiangular to the triangle

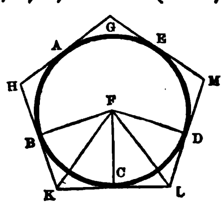
FGH by constⁿ, and each of the angles G, H is double of the angle F; therefore each of the angles ACD, ADC is double of the angle CAD, and they are bisected by CE, DB: hence the five angles ADB, BDC, CAD, ECD, ECA are all equal. But in the same circle equal angles at the circumference stand on equal arcs (iii. 26); therefore the five arcs, AB, BC, CD, DE, EA are all equal: and in the same circle equal arcs are subtended by equal straight lines (iii. 29); therefore the five straight lines AB, BC, CD, DE, EA are all equal, and the pentagon ABCDE is equilateral. Again, because the arc AB is equal to the arc DE; to each of these equals add the arc BCD: then the whole arc ABCD is equal (Ax. 2) to the whole arc EDCB. But in the same circle the angles at the circumference standing on equal arcs are equal (iii. 27); therefore the angle BAE is equal to the angle AED. In like manner it may be shewn that each of the angles ABC, BCD, CDE is equal to the angle BAE or AED: therefore the five angles are all equal and the pentagon equiangular. Hence the pentagon ABCDE is both equilateral and equiangular, and is therefore a regular pentagon; and since all its angular points A, B, C, D, E are in the circumference, it is inscribed (iv. Def. 3) in the given circle ABC. Which was to be done.

PROP. XII. PROB.

To circumscribe a regular pentagon about a given circle.

Let ABC be the given circle. It is required to circumscribe a regular pentagon about the circle ABC.

In the circle ABC inscribe (iv. 11) a regular pentagon, and let its angular points be A, B, C, D, E, which lie in the circumference and through A, B, C, D, E draw (iii. 17) GH, HK, KL, LM, MG touching the circle ABC, and meeting one another so as to form the five-sided figure GHKLM. Then GHKLM shall be the pentagon reguired.



Find (iii. 1) the centre F of the circle; and join FB, FK, FC, FL, FD.

Then because A, B, C, D, E are

the angular points of the inscribed regular pentagon; and in the same circle equal straight lines cut off equal arcs (iii. 28): therefore the arcs AB, BC, CD, DE, EA, cut off by the sides of the inscribed regular pentagon are equal. And because KL touches the circle by constn, and FC is drawn from the centre F to the point of contact C, FC is perpendicular (iii. 18) to KL, and each of the angles at C is a right angle: for like reason each of the angles at B, D is a right angle. Now because FCK is a right angle, the squares of FC, CK are equal (i. 47) to the square of FK, and because FBK is a right angle, the squares of FB, BK are equal to the square of FK; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the squares of FC, CK are equal to the squares of FB, BK. But the square of FC is equal to the square of FB, because FC is equal to FB by the defn of a circle: hence taking away equals from equals, the remaining square of CK is equal (Ax. 3) to the remaining square of BK, and therefore CK is equal to BK. Also FC is equal to FB, and FK common to the two triangles FCK, FBK; therefore these two triangles have the three sides FC, CK, KF respectively equal to the three sides FB, BK, KF. Therefore they are equal in every respect (i. 8); and hence the angle CKF is equal to the angle BKF, and the angle CFK to the angle BFK. That is, FK bisects each of the angles BKC, BFC; and it may be shewn in like manner that FL bisects each of the angles CLD, CFD:

Again, because the arc BC is equal to the arc CD, and in the same circle the angles at the centre standing on equal arcs are equal (iii. 27), the angle BFC is equal to the angle CFD; and the halves of equal things are equal (Ax. 7): therefore the angle KFC is equal to the angle LFC. Also the angle KCF is equal to the angle LCF (Def. 10), because FC is perpendicular to KL, and FC is common to the two triangles FCK, FCL; therefore these two triangles have the two angles KFC, FCK respectively equal to the two angles LFC, FCL, and the side FC, adjacent to equal angles in each, common. Therefore they are equal in every respect (i. 26); and hence the side KC is equal to the side CL, and the angle FKC equal to the angle FLC.

Now because KC is equal to KL, KL is double of KC; and it may be shewn in like manner that HK is double of BK; but BK was proved equal to KC; and the doubles of equal things are equal (Ax. 6): therefore HK is equal to KL. Similarly it may be proved that the sides including each of the other angles of the figure are equal: therefore the five sides HK, KL, LM, MG, GH are all equal, and the pentagon GHKLM is equilateral:

Lastly, because the angle BKC is double of the angle FKC, and the angle CLD of the angle FLC, and the angles FKC, FLC are equal: therefore, since the doubles of equal things are equal, the angle HKL is equal to the angle KLM. Similarly it may be shewn that each adjacent pair of the angles of the figure are equal: therefore the five angles at G, H, K, L, M are all equal, and the pentagon GHKLM is equiangular. Hence the pentagon GHKLM is both equilateral and equiangular, and is therefore a regular pentagon; and since each of its sides touches the given circle ABC, it is circumscribed (IV. Def. 4) about it. Which was to be done.

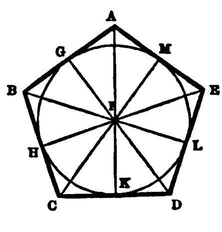
PROP. XIII. PROB.

To inscribe a circle in a given regular pentagon.

Let ABCDE be the given regular pentagon. It is required to inscribe a circle in the regular pentagon ABCDE.

Bisect (I. 9) any two adjacent angles BCD, CDE of the pentagon by the straight lines CF, DF, cutting one another in F: and from F draw (I. 12) FG, FH, FK, FL, FM perpendicular to the sides of the pentagon AB, BC, CD, DE, EA.

Join FA, FB, FC, FD, FE.



Because BC is equal to CD, since they are sides of the regular pentagon; CF common to the two triangles BCF, DCF; and the angles BCF, FCD equal by constn: therefore these two triangles have the two sides BC, CF respectively equal to the two sides DC, CF, and the included angle BCF equal to the included angle DCF. Therefore they are equal in every respect

(i. 4); and hence the angle CBF is equal to the angle CDF. But the angle CDF is half of the angle CDE by constn, and the angle CDE is equal to the angle CBA, since they are angles of the regular pentagon: therefore the angle CDF is half of the angle CBA. But the angle CBF was shewn to be equal to the angle CDF; and things that are equal to the same thing are equal to one another (Ax. 1): therefore likewise the angle CBF is half of the angle CBA, that is, BF bisects the angle CBA. In like manner it may be proved that FA bisects the angle BAE, and FE the angle AED:

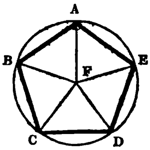
Now because the angle FCH is equal to the angle FCK; and the right angle FHC to the right angle FKC, because all right angles are equal to one another (Ax. 11); and FC is common to the two triangles FCH, FCK: therefore these two triangles have the two angles FHC, FCH respectively equal to the two angles FKC, FCK, and the side FC, opposite to equal angles in each, common. Therefore they are equal in every respect (i. 26); and hence FH is equal to FK. Similarly it may be proved that FK is equal to FL, FL to FM, FM to FG, FG to FH: therefore the five straight lines FG, FH, FK, FL, FM are all equal. Hence the circle described with centre F and any one of them as radius will pass through the extremities of the other four; and since each of the straight lines AB, BC, CD, DE, EA touch it, because the angles at G, H, K, L, M are right angles, and the straight line drawn perpendicular to a radius of a circle from its extremity touches the circle (iii. 16, Cor.), therefore the circle GHK is inscribed (iv. Def. 5) in the given regular pentagon ABCDE. Which was to be done.

PROP. XIV. PROB.

To circumscribe a circle about a given regular pentagon.

Let ABCDE be the given regular pentagon. It is required to circumscribe a circle about the regular pentagon ABCDE.

Bisect (i. 9) any two adjacent angles BCD, CDE of the pentagon by the straight lines CF, DF, cutting one another in F; and join FB, FA, FE.



Then it may be proved as in the preceding prop^n that the angles CBA, BAE, AED are bisected respectively by FB, FA, FE. And because the angle BCD is equal to the angle CDE, since they are angles of a regular pentagon; and the halves of equal things are equal (Ax. 7) : therefore the angle FCD is equal to the angle FDC; and therefore FC is equal (i. 6) to FD. In like manner it may be shewn that each other adjacent pair of the five straight lines FA, FB, FC, FD, FE are equal; therefore they are all equal. Hence the circle described with centre F and any one of them as radius will pass through the extremities of the other four; and since it passes through all the angular points A, B, C, D, E, the circle ABC is circumscribed (iv. Def. 6) about the given regular pentagon ABCDE. Which was to be done.

PROP. XV. PROB.

To inscribe a regular hexagon in a given circle.

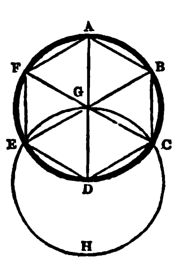
Let ABC be the given circle. It is required to inscribe a regular hexagon in the circle ABC.

Find (iii. 1) the centre G, of the circle ABC, and draw any diameter AGD. With centre D and radius DG describe the circle EGCH, cutting the circle ABC in E and C; join EG, CG, and produce them to cut the circle ABC in B and F. Join AB, BC, CD, DE, EF, FA. Then the six-sided figure ABCDEF shall be the regular hexagon required.

Because G is the centre of the circle ABC, GE is equal to GD by def^n, and because D is the centre of the circle CGE, DE is equal to DG for the same reason; and things that are equal to the same thing are equal to one another (Ax. 1): therefore GE is equal to ED, and the three straight lines GE, EG, DG are equal. Therefore the triangle GED is equilateral; and therefore it is also equiangular (i. 5. Cor.). But its three angles GED, EDG, DGE are equal (i. 32) to two right angles; therefore any one of them EGD is equal to the third part of two right angles. In like manner it may be proved that the angle DGC is

the third part of two right angles. Hence the whole angle EGC is equal to two thirds of two right angles; but because CG makes with EGB on the same side of it the adjacent angles EGC, CGB, these two angles are equal (i. 13) to two right angles, and one of them EGC is equal to two thirds of two right angles: therefore the other angle CGB is equal to one third of two right angles. Hence the three angles EGD, DGC, CGB are equal; and to these are equal the opposite vertical angles BGA, AGF, FGE (i. 15): therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are all equal. But in the same circle equal angles at the centre stand on equal arcs (iii. 26); therefore the six arcs AB, BC, CD, DE, EF, FA are all equal: and in the same circle equal arcs are subtended by equal straight lines (iii. 29), therefore the six straight lines AB, BC, CD, DE, EF, FA are all equal, and the hexagon ABCDEF is equilateral. Again because the arc AF is equal to the arc ED; to each of these equals add the arc ABCD: then the whole arc FABCD is equal (Ax. 2) to the whole arc EDCBA. But in the same circle the angles at the circumference standing on equal arcs are equal (iii. 27); therefore the angle AFE is equal to the angle FED. In like manner it may be shewn that each other adjacent pair of the angles of the figure are equal: therefore the six angles are all equal and the hexagon equiangular. Hence the hexagon ABCDEF is both equilateral and equiangular, and is therefore a regular hexagon; and since all its angular points A, B, C, D, E, F are in the circumference of the given circle ABC, it is inscribed (iv. Def. 3) in it. Which was to be done.

COR.—The side of a regular hexagon inscribed in a circle shall be equal to the radius of the circle.



For it was shewn in the proof of the prop[n] that GE is equal to ED, and ED is the side of the inscribed regular hexagon, and GE the radius of the circle. Therefore the side of the in-

scribed regular hexagon is equal to the radius of the circle. Which was to be proved.

OBS.—If through A, B, C, D, E, F, the angular points of the inscribed regular hexagon, there be drawn straight lines touching the circle, these shall form a regular hexagon circumscribing the circle; and also if two adjacent angles of a regular hexagon be bisected, and from the point where the bisecting line meets perpendiculars be drawn to the sides of the hexagon, then the circle described with this point as centre and any one of the perpendiculars as radius, shall be the circle inscribed in the regular hexagon.

The proof of these is precisely similar to those of Propns 12 and 13. In fact, the three Propns 12, 13, 14 may be generalized so as to apply to any regular polygon; care being taken to substitute polygon for pentagon throughout, and the propositions being in other respects proved alike. The general form of enunciation is as follows :—

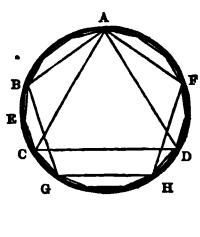
(1) A regular polygon may be circumscribed about a given circle by drawing straight lines touching the circle through the angular points of the regular inscribed polygon of the same number of sides.

(2) A circle may be inscribed in a regular polygon by bisecting two of its adjacent angles, drawing perpendiculars on its sides from the point where the bisecting lines meet, and describing a circle with this point as centre and any one of the perpendiculars as radius.

(3) A circle may be circumscribed about a regular polygon by bisecting two of its adjacent angles, joining the point where the bisecting lines meet with the angular points of the polygon, and describing a circle with this point as centre and any one of the joining lines as radius.

PROP. XVI. PROB.

To inscribe a regular polygon of fifteen sides, or quindecagon in a given circle.



Let ABC be the given circle. It is required to describe a regular quindecagon in the circle ABC.

Describe an equilateral and therefore an equiangular (i. 5, Cor.) triangle, and inscribe (iv. 2) in the circle ABC a triangle ACD equiangular to it; then the triangle ACD will be an equiangular triangle, and

therefore equilateral (i. 6, Cor.). Inscribe (iv. 11) also in the circle a regular pentagon ABGHF, having one of its angular points A coinciding with one of those of the triangle ACD.

Since in the same circle equal straight lines cut off equal arcs (iii. 28), the three arcs AC, CD, DA are all equal, and therefore the arc AC is a third part of the whole circumference of the circle: similarly it may be shewn that the arc AB is a fifth part of it. Hence of the fifteen equal arcs into which the whole circumference may be divided, the arc AC contains five, and the arc AB three, and consequently their difference, the arc BC, contains two. Bisect (iii. 30) the arc BC in E; therefore each of the arcs BE, EC is a fifteenth part of the whole circumference. Hence if the straight lines BE, EC be joined, and straight lines equal to them be placed (iv. 1) around in the whole circle, contiguous to one another, and in number fifteen, they will form one equilateral quindecagon inscribed in the circle ABC; and this equilateral quindecagon may be shewn to be equiangular, and therefore regular, as the equilateral pentagon was in the concluding part of the proof of Prop. XI. Hence in the given circle ABC has been inscribed a regular quindecagon. Which was to be done.

THE

ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is defined to be a measure or submultiple of a greater magnitude of the same kind, when the less magnitude is contained a certain number of times exactly in the greater.

OBS. Such a less magnitude is sometimes called a "part" of the greater; but when part is thus used as equivalent to measure or submultiple, we must be careful not to confound this restricted signification of part with the general one employed in Book I. Ax. 9.

II.

A greater magnitude is defined to be a multiple of a less magnitude of the same kind, when the greater magnitude contains the less a certain number of times exactly.

OBS. When two or more greater magnitudes contain two or more less magnitudes of corresponding kinds the same number of times exactly, the former magnitudes are called "equimultiples" of the latter.

III.

Ratio is a mutual relation of two magnitudes of the same kind to one another in respect of quantity.

Obs. It appears that for one magnitude to have a ratio to another, they must both be of the same kind. Although in this and the following def^{n} Euclid lays down the conditions for two magnitudes to have a ratio to one another, he nowhere gives a mode of estimating a single ratio; he defines only when one ratio is the same as (Def. 5), greater than, or less than (Def. 7) another ratio.

IV.

Two unequal magnitudes of the same kind are said to have a ratio to one another, provided that the less can be multiplied a sufficient number of times for its multiple to exceed the greater; and two equal magnitudes of the same kind are said to have a ratio to one another, and it is called a ratio of equality.

Obs. This def^{n} excludes from our present consideration all ratios between (1) two magnitudes, both of which are infinitely great or infinitely small; (2) two magnitudes, one of which is finite, and the other infinitely great or infinitely small.

V.

The first of four magnitudes is defined to have the same ratio to the second which the third has to the fourth, when, there having been taken of the first and third any equimultiples whatever, and of the second and fourth any equimultiples whatever, the multiple of the third is greater than, equal to, or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.

Obs. 1. The word "any" in the def^{n} must be specially noted. In the application of this def^{n} to test the proportionality (Def. 6) of four magnitudes, the criterion must be satisfied in every instance for them to be proportional, whereas a single failure would shew their non-proportionality.

Obs. 2. It appears from Def. 3, that the first and second of the four magnitudes must be of the same kind, and also that the third and fourth must be of the same kind. But there is no necessity for all four to be of the same kind.

Obs. 3. When the first of four magnitudes has the same ratio to the second which the third has to the fourth, the third clearly has the same ratio to the fourth which the first has to the second. Such will appear also from the def^{n}.

Obs. 4. When this def^{n} is referred to, it is cited as the "definition of proportion."

VI.

When the first of four magnitudes has the same ratio to the second which the third has to the fourth (such sameness of ratios being defined by the preceding def^n), the four magnitudes are said to constitute a proportion, or to be proportional.

Obs. 1. When four magnitudes, A, B, C, D, are proportional, the proportion which they constitute is usually expressed by saying that A is to B as C to D, and may be written

A : B : : C : D.

But whenever either of these forms is used to denote a proportion, we must bear in mind that it is but a brief way of stating that A has the same ratio to B which C has to D, according to the def^n of sameness of ratio given in Def. 5.

Obs. 2. When four magnitudes are proportional :—

(1) they are called the first, second, third, and fourth terms of the proportion;

(2) the fourth term is called a fourth proportional to the three magnitudes which form the first, second, and third terms;

(3) the first and fourth terms are called the two extremes, and the second and third the two means.

VII.

The first of four magnitudes is defined to have a greater ratio to the second than the third has to the fourth (or, what is the same thing, the third to have a less ratio to the fourth than the first has to the second) when there can be taken such equimultiples of the first and third, and such equimultiples of the second and fourth, that the multiple of the first shall be greater than the multiple of the second, but the multiple of the third not greater than the multiple of the fourth.

Obs. When this def^n is referred to, it is cited as the "definition of one ratio being greater (or less) than another."

VIII.

By proportion is meant the sameness of ratios.

Obs. This is nothing more than a different way of expressing Def. 6.

IX.

In order that four magnitudes may constitute a proportion, no more than two of them are allowed to be equal to the same magnitude.

OBS. 1. Hereby Euclid excludes from consideration all such proportions as A is to A as A to B, A is to B as B to B, &c. But we shall meet with such proportions as A is to B as A to B, A is to A as B to B (Def. 4), where two pairs of the four magnitudes are equal to the same magnitude; and with such as A is to B as C to A, A is to B as B to C, where one pair of the four are equal to the same magnitude. Since proportions like the last frequently occur, we shall speak of them in the next observation.

OBS. 2. When of three different magnitudes of the same kind, one can form the first term of a proportion, another each of the second and third terms, and the remaining magnitude the fourth term; i.e. if the first of the three magnitudes be to the second as the second is to the third: then

(1) the three magnitudes are said to be proportional;

(2) the third magnitude is said to be a third proportional to the first and second;

(3) the second magnitude is said to be a mean proportional between the first and third.

X.

The duplicate ratio of the ratio which one magnitude has to another is defined to be the ratio which the first magnitude has to a third magnitude such that the first is to the second as the second is to the third.

OBS. 1. Hence when three magnitudes are proportional (Def. 9. Obs. 2), the ratio which the first bears to the third is the duplicate ratio of the ratio which it bears to the second.

OBS. 2. This definition will be made clearer by the following example:—

Ex. It is required to find the duplicate ratio of the ratio of A to B.

Take a magnitude C of the same kind as A and B, such that A is to B as B to C. Then the ratio of A to C will be the duplicate ratio of the ratio of A to B.

XI.

When there are any number of magnitudes of the same kind, the ratio of the first of them to the last

is defined to be the ratio compounded of the ratios of the first to the second, of the second to the third, of the third to the fourth, &c., and of the last but one to the last.

OBS. 1. In the particular case when the ratios of the first magnitude to the second, of the second to the third, &c., are all the same one to another, the ratio compounded of them as above defined is called a multiplicate ratio of either of the component ratios, the order of multiplicity being equal to their number.

The def^n and the observation will be made clearer by the following examples:—

Ex. 1. If there are three magnitudes of the same kind, A, B, C; the ratio of A to C is the ratio compounded of the two ratios of A to B, and of B to C. And in the particular case when the ratio of A to B is the same as that of B to C, the ratio of A to C is called the duplicate ratio of that of A to B, or of B to C.

Ex. 2. If there are four magnitudes of the same kind, A, B, C, D; the ratio of A to D is the ratio compounded of the three ratios of A to B, and of B to C, and of C to D. And in the particular case when the ratios of A to B, of B to C, and of C to D are the same to one another, the ratio of A to D is called the triplicate ratio of that of A to B, or of B to C, or of C to D.

OBS. 2. The ratios which this def^n, as it stands, enables us to compound, must consist of magnitudes of the same kind, and the second term of each ratio must be the same as the first of the succeeding one. The manner of modifying the def^n to compound any ratios whatever will be seen from the annexed example.

Ex. Let there be any three ratios whatever; viz. the ratio of A to B, the ratio of C to D, and the ratio of E to F. It is required to find the ratio compounded of these three ratios.

Take two magnitudes of any kind (usually straight lines) K, L, such that A is to B as K to L; take a magnitude M of the same kind as K and L, such that C is to D as L to M; and take a magnitude N of the same kind as K, L, M, such that E is to F as M to N. Then the ratio compounded of the three ratios of A to B, of C to D, and of E to F, is the same as the ratio compounded of the three ratios of K to L, of L to M, and of M to N; but the ratio compounded of the three ratios of K to L, of L to M, and of M to N is by def^n the ratio of K to M. Hence the ratio of K to M is the ratio compounded of the three ratios of A to B, of C to D, and of E to F. And a similar method must be employed whatever be the number of ratios to be compounded.

XII.

Of two magnitudes, having a ratio to one another, the first is called the antecedent, and the second the

consequent; and of four magnitudes constituting a proportion, the first and third are called the antecedents, and said to be homologous to one another, and the second and fourth are called the consequents, and are likewise said to be homologous to one another.

POSTULATES.

I.

Let it be granted that of a given magnitude there may be taken any multiple required.

II.

Let it be granted that any given multiple of a magnitude may be divided into parts, each of which is equal to the magnitude of which it is the multiple.

AXIOMS.

I.

Equimultiples of the same magnitude, or of equal magnitudes, are equal to one another.

II.

Magnitudes, of which the same magnitude is an equimultiple, or of which equal magnitudes are equimultiples are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less magnitude.

IV.

Of two magnitudes, the first is greater or less than the second according as the multiple of the first is greater or less than the same multiple of the second.

PROPOSITIONS.

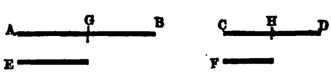
PROP. I. THEOR.

If a set of magnitudes be equimultiples of the same number of other magnitudes: then, whatever multiple either one in the first set is of the corresponding one in the second, the same multiple shall all the first set of magnitudes taken together be of all the second set of magnitudes taken together.

I. Let the number of the magnitudes in each set be two.

Let the two magnitudes AB, CD be respectively equimultiples of the two E, F. Then whatever multiple AB is of E, the same multiple shall AB and CD together be of E and F together.

Since by hyp^{s} AB, CD are equimultiples of E, F, there are as many magnitudes in AB, each equal to E, as there are in CD, each equal to F. Divide (v. Post. 1) AB into magnitudes, each equal to E, viz. AG, GB, and CD into magnitudes, each equal to F, viz. CH, HD; the number of AG, GB being equal to that of CH, HD.



By const^{n} AG is equal to E, and CH to F; hence, adding equals to equals, AG and CH together are equal (Ax. 3) to E and F together. Similarly GB and HD together are equal to E and F together: and so on, if there were more magnitudes in AB and CD. Hence there are as many

magnitudes in AB and CD together, each equal to E and F together, as there are in AB, each equal to E; that is, whatever multiple (v. Def. 2) AB is of E, the same multiple are AB and CD together of E and F together. Which was to be proved.

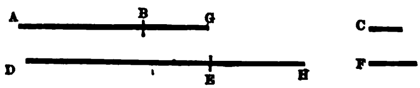
II. Let the number of magnitudes in each set be greater than two.

The same method of proof that has been used in Case I. for the number two, can be applied to any number.

PROP. II. THEOR.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth: then shall the first and the fifth together be the same multiple of the second that the third and the sixth together are of the fourth.

Let AB be the same multiple of C that DE is of F, and BG the same multiple of C that EH is of F. Then shall AB and BG together, that is, AG be the same multiple of C that DE and EH together, that is, DH, is of F.



Since by hyp[s] AB, DE are equimultiples of C, F, there are as many (v. Def. 2. Obs.) magnitudes in AB, each equal to C, as there are in DE each equal to F. Similarly there are as many magnitudes in BG, each equal to C, as there are in EH, each equal to F. Hence, adding equal numbers to equal, there are as many (Ax. 2) magnitudes in AG, each equal to C, as there are in DH, each equal to F; that is, AG is the same multiple (v. Def. 2) of C that DH is of F Which was to be proved.

COR.—If a set of magnitudes be multiples of one magnitude, and the same number of other magni-

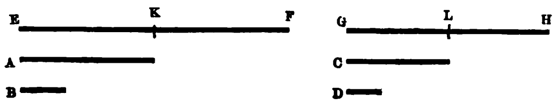
tudes be multiples of another, each of the first set being the same multiple of the one as a corresponding one in the second set is of the other: then shall all the first set of magnitudes together be the same multiple of the one magnitude as all the second set together are of the other.

The proof of this is exactly similar to the one applied in the propⁿ to the case of the number of magnitudes in each set being two.

PROP. III. THEOR.

If the first magnitude be the same multiple of the second that the third is of the fourth: then any equimultiples of the first and third shall likewise be equimultiples of the second and fourth.

Let A be the same multiple of B that C is of D, and let EF, GH be any equimultiples of A, C. Then EF, GH shall likewise be equimultiples of B, D.



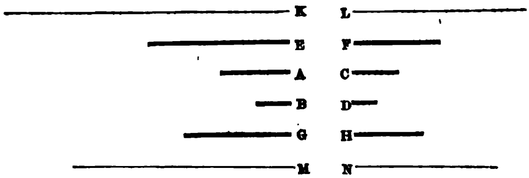
Since by hyp[s] EF is the same multiple of A that GH is of C, there are as many (v. Def. 2) magnitudes in EF, each equal to A, as there are in GH, each equal to C. Divide (v. Post. 2) EF into the magnitudes EK, KF, each equal to A, and GH into the magnitudes GL, LH, each equal to C; the number of EK, KF being equal to that of GL, LH.

By hyp[s] A is the same multiple of B that C is of D; and by const[n] EK is equal to A, and GL to C: therefore EK is the same multiple of B that GL is of D. Similarly KF is the same multiple of B that LH is of D; and so on, if there are more magnitudes in EF and GH. Hence (v. 2, Cor.) EK, KF together are the same multiple of B that GL, LH together are of D; that is, EF, GH are equimultiples of B, D. Which was to be proved.

PROP. IV. THEOR.

If four magnitudes be proportional, and there be taken of the first and third any equimultiples whatever, and of the second and fourth any equimultiples whatever: then the multiple of the first shall be to that of the second as the multiple of the third is to that of the fourth.

Let the four magnitudes A, B, C, D be proportionals; and let there be taken of the first A and third C any equimultiples whatever E, F and of the second B and fourth D any equimultiples whatever G, H. Then shall E be to G as F is to H.



Of E, F take (v. Post. 1) any equimultiples whatever K, L; and of G, H any equimultiples whatever M, N.

By hyp^{s} E is the same multiple of A that F is of C, and by const^{n} K is the same multiple of E that L is of F; therefore K is the same multiple (v. 3) of A that L is of C. In like manner M is the same multiple of B that N is of D. But A is to B as C to D by hyp^{s}, and there have been taken of the first A and third C equimultiples K, L, and of the second B and fourth D equimultiples M, N; therefore by the def^{n} of proportion (v. Def. 5) L is greater than, equal to, or less than N, according as K is greater than, equal to, or less than M. Hence, since there are four magnitudes E, G, F, H, and there have been taken of the first E and third F any equimultiples whatever K, L, and of the second G and fourth H any equimultiples whatever M, N; and since it has been shewn that the multiple of the third L is greater than, equal to, or less than that of the fourth N according as the multiple of the first K is greater than, equal to, or less than that of the second M: there-

fore by the def^n of proportion, E is to G as F to H. Which was to be proved.

COR.—If four magnitudes be proportional: then

(1) any equimultiples whatever being taken of the first and third, the multiple of the first shall be to the second as the multiple of the third is to the fourth;

(2) any equimultiples whatever being taken of the second and fourth, the first shall be to the multiple of the second as the third is to the multiple of the fourth.

Let the four magnitudes A, B, C, D be proportional. Then:—

(1) if any equimultiples whatever E, F be taken of A and C, E shall be to B as F is to D.

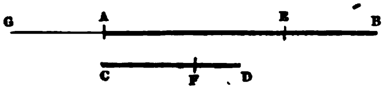
(2) if any equimultiples whatever G, H be taken of B and D, A shall be to G as C is to H.

Both parts of the Corollary are proved by a method similar to that employed in the prop^n; the const^n for (1) being to take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H; and that for (2) being to take of A, C any equimultiples whatever E, F, and of G, H any equimultiples whatever M, N.

PROP. V. THEOR.

If the first magnitude be the same multiple of the second that a third magnitude of the same kind is of the fourth, the first being greater than the third, and the second than the fourth: then the excess of the first above the third shall be the same multiple of that of the second above the fourth that the first is of the second.

Let AB be the same multiple of CD that AE a magnitude of the same kind is of CF; AB being greater than AE, and consequently CD than CF. Then EB, the excess of AB above AE, shall be the same multiple of FD, the excess of CD above CF, that AB is of CD.



Of FD take AG the same multiple that AE is of CF.

Since by constᵃ AE, AG are equimultiples of CF, FD, AE and AG together, that is, EG is the same multiple (v. 1) of CF and FD, together, that is, CD, that AE is of CF; but by hypᵉ AE is the same multiple of CF that AB is of CD: therefore EG is the same multiple of CD that AB is of CD. And equimultiples of the same magnitude are equal (v. Ax. 1); therefore EG is equal to AB. From each of these equals take away the common magnitude AE; then the remainder AG is equal (Ax. 3) to the remainder EB. And AE is the same multiple of CF that AG is of FD; hence AE is the same multiple of CF that EB is of FD. But AE is the same multiple of CF that AB is of CD; therefore EB is the same multiple of FD that AB is of CD. Which was to be proved.

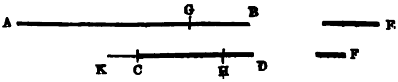
PROP. VI.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two: then the remainders shall either be equal to these others, or equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F; and let AG, CH taken from AB, CD be also equimultiples of E, F. Then the remainders GB, HD shall be either equal to E, F, or else equimultiples of them.

Since AB, AG are multiples of E, and can therefore be each of them divided into a number of magnitudes each equal to E, it is clear that the remainder GB is either equal to E, or a multiple of E.

I. Let GB be equal to E.

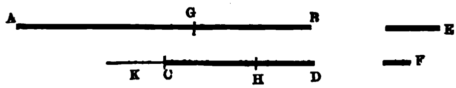


Take CK equal to F.

Because by hyp^s AG is the same multiple of E that CH is of F; and BG is supposed equal to E, and CK by const^n is equal to F: therefore AB is the same multiple of E that KH is of F. But by hyp^s AB is the same multiple of E that CD is of F; therefore KH is the same multiple of F that CD is of F: and equimultiples of the same magnitude are equal (V. Ax. 1): therefore KH is equal to CD. From each of these equals take away the common magnitude CH; then the remainder KC is equal (Ax. 3) to the remainder HD. But KC is equal to F; and things that are equal to the same thing are equal to one another (Ax. 1): therefore HD is equal to F.

II. Let GB be a multiple of E.

Take CK the same multiple of F that GB is of E.



Because by hyp^s AG is the same multiple of E that CH is of F; and by const^n GB the same multiple of E that CK is of F: therefore AB is the same multiple of E that KH is of F. Hence as in Case I. it may be shewn that KC is equal to HD; and KC is the same multiple of F that GB is of E: therefore likewise HD is the same multiple of F that GB is of E.

Hence the remainders GB, HD are either equal to E, F, or else equimultiples of them. Which was to be proved.

PROP. A. THEOR.

If four magnitudes be proportional; then the third shall be greater than, equal to, or less than the fourth according as the first is greater than, equal to, or less than the second.

Of all four take any equimultiples as the doubles.

Then since the four magnitudes are proportional, and there have been taken of the first and third equimultiples, viz. the doubles, and of the second and fourth equimultiples, viz. the doubles: therefore by the def^n of proportion (V. Def. 5), the double of the third is greater than, equal to, or

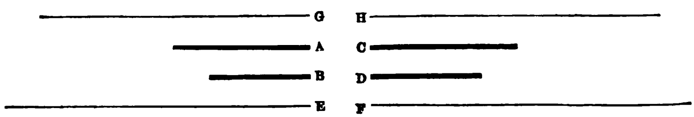
less than that of the fourth, according as the double of the first is greater than, equal to, or less than that of the second. But the double of the third is greater than, equal to, or less than that of the fourth, according as the third is greater than, equal to, or less than the fourth: and so, mutatis mutandis, for the first and second: hence the third is greater than, equal to, or less than the fourth, according as the first is greater than, equal to, or less than the second. Which was to be proved.

PROP. B. THEOR.

If four magnitudes be proportional: then they shall also, when taken inversely, be proportionals, i. e. the second shall be to the first as the fourth is to the third.

Obs. This proposition is usually cited by the word "invertendo."

Let A be to B as C is to D. Then likewise B shall be to A as D is to C.



Of B, D take any equimultiples whatever, E, F, and of A, C any equimultiples whatever, G, H.

Because by hyp^{s} A is to B as C to D, and there have been taken of the first A and third C equimultiples G, H, and of the second B and fourth D equimultiples E, F: therefore by the def^{n} of proportion (v. Def. 5), H is greater than, equal to, or less than F, according as G is greater than, equal to, or less than E. Hence manifestly F is greater than, equal to, or less than H, according as E is greater than, equal to, or less than G. Now since there are four magnitudes B, A, D, C, and there have been taken of the first B and third D any equimultiples whatever E, F, and of the second A and fourth C any equimultiples whatever G, H; and since it has been shewn that the multiple of the third F is greater than, equal to, or less than that of the fourth H, according as the multiple of the first

E is greater than, equal to, or less than that of the second G: therefore by the def^n of proportion, B is to A as D to C. Which was to be proved.

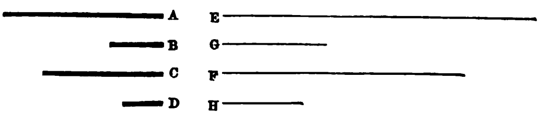
PROP. C. THEOR.

If the first of four magnitudes be either the same multiple or the same part (i. e. measure or submultiple) of the second that the third is of the fourth: then the four magnitudes shall be proportional.

Let A, B, C, D be the four magnitudes. Then there are two cases, according as A, C respectively are multiples or parts of B, D.

I. Let A be the same multiple of B that C is of D. Then shall A be to B as C is to D.

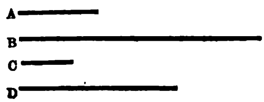
Of A, C take any equimultiples whatever E, F; and of B, D any equimultiples whatever G, H.



Then because A, C are by hyp^s equimultiples of B, D; and E, F by const^n are equimultiples of A, C: E, F likewise are equimultiples (V. 3) of B, D. Also G, H by const^n are equimultiples of B, D. Hence F is a greater, equal, or less multiple of D than H is, according as E is a greater, equal, or less multiple of B than G is; and therefore F is greater than, equal to, or less than H, according as E is greater than, equal to, or less than G. Now since there are four magnitudes, A, B, C, D, and there have been taken of the first A and third C any equimultiples whatever E, F, and of the second B and fourth D any equimultiples whatever G, H; and since it has been shewn that the multiple of the third F is greater than, equal to, or less than that of the fourth H, according as the multiple of the first E is greater than, equal to, or less than that of the second G: therefore by the def^n of proportion

(v. Def. 5), A is to B as C to D. Which was to be proved.

II. Let A be the same part, that is, measure or submultiple (v. Def. 1. Obs.) of B that C is of D. Then shall A be to B as C is to D.



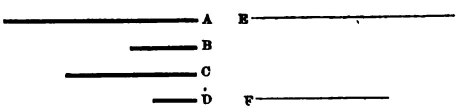
From the defⁿ of part (v. Def, 1) and that of multiple (v. Def. 2), it appears that B will be the same multiple of A that D is of C. Therefore by Case I. of the propⁿ, since B, A, D, C are four magnitudes, and B is the same multiple of A that D is of C, B is to A as D to C. Hence, invertendo (v. B), A is to B as C to D. Which was to be proved.

PROP. D. THEOR.

If four magnitudes be proportional, and the first be either a multiple or a part (i. e. measure or submultiple) of the second: then the third shall be either the same multiple or the same part of the fourth.

Let A be to B as C is to D. Then there are two cases of the propⁿ, according as A is a multiple or a part of B.

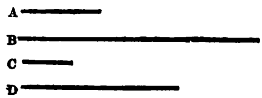
I. Let A be a multiple of B. Then C shall be the same multiple of D.



Take E equal to A; and whatever multiple A or E is of B, take F the same multiple of D.

By hypˢ A is to B as C to D; and there have been taken of the second B and fourth D equimultiples E, F: therefore (v. 4, Cor.) A is to E as C to F. But A is equal to E by constⁿ; therefore also (v. A) C is equal to F. Now by constⁿ F is the same multiple of D that A is of B; hence C is the same multiple of D that A is of B. Which was to be proved.

II. Let A be a part, that is, measure or submultiple (v. Def. I. Obs.) of B. Then C shall be the same part of D.



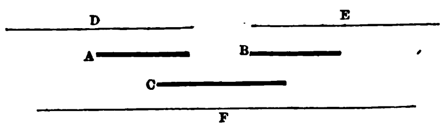
Since A is supposed a part of B, it appears by the def^{n} of part (v. Def. I) and that of multiple (v. Def. 2) that B is a multiple of A. Now A is to B as C to D, therefore, invertendo (v. B), B is to A as D to C; and it has been shewn that B is a multiple of A: therefore by Case I. of the prop^{n} D is the same multiple of C that B is of A. Hence C is the same part of D that A is of B. Which was to be proved.

PROP. VII. THEOR.

Two equal magnitudes shall have each of them the same ratio to a third magnitude of the same kind. And to each of two equal magnitudes a third magnitude of the same kind shall have the same ratio.

Let A, B be two equal magnitudes, and C any other magnitude of the same kind. Then:—

I. A and B shall have each of them the same ratio to C; that is, A shall be to C as B is to C.



Of A, B take any equimultiples whatever D, E; and of C any multiple whatever F.

By const^{n} D, E are equimultiples of A, B, which are equal by hyp^{s}; and equimultiples of equal magnitudes are equal (v. Ax. I): therefore D is equal to E. Now since there are four magnitudes A, C, B, C, and there have been taken of the first A and third B any equimultiples whatever D, E, and of the second and fourth C any multiple whatever F; and since the multiple of the third E is greater than, equal to, or less than that of the fourth F,

according as the multiple of the first D is greater than, equal to, or less than that of the second F, for D has been shewn to be equal to E: therefore by the def^{n}. of proportion (v. Def. 5), A is to C as B to C. Which was to be proved.

II. C shall have the same ratio to each of A and B; that is, C shall be to A as C is to B.

Construct as in Part I. of the prop^{n}.

Then as before it may be shewn that D is equal to E. Now since there are four magnitudes C, A, C, B, and there have been taken of the first and third C any multiple whatever F, and of the second A and fourth B any equimultiples whatever D, E; and since the multiple of the first F is greater than, equal to, or less than that of the second D according as the multiple of the third F is greater than, equal to, or less than that of the fourth E, for D is equal to E: therefore by the def^{n} of proportion C is to A as C to B. Which was to be proved.

PROP. VIII. THEOR.

The greater of two unequal magnitudes shall have a greater ratio to a third magnitude of the same kind than the less has to it. And to the less of two unequal magnitudes a third magnitude of the same kind shall have a greater ratio than it has to the greater.

Let AB be the greater, and BC the less of two unequal magnitudes; and let D be any other magnitude of the same kind. Then:—

I. AB shall have a greater ratio to D than BC has to D.

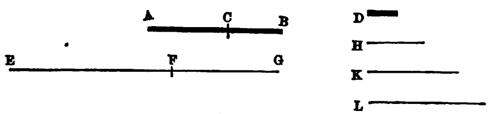


Fig. 1.

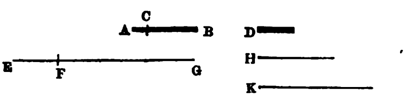


Fig. 2.

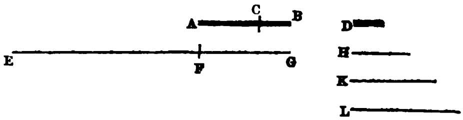


Fig. 3.

Then AC being the other part of the greater magnitude AB, which together with the part BC makes up the whole AB, that which is not the greater of the two magnitudes AC, BC must either be not less than the magnitude D, or less than it. If the magnitude which is not the greater of the two AC, CB be not less than D (Fig. 1), take EF, FG the doubles of AC, CB. But if the magnitude which is not the greater of the two AC, CB be less than D (Figs 2 and 3), this magnitude can be multiplied a sufficient number of times for its multiple to exceed D (v. Def. 4), whether it be AC (Fig. 2), or BC (Fig. 3). Let it be multiplied until its multiple exceeds D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the same multiple of CB. Therefore in all the three figures EF, FG are each of them greater than D; and in all the three figures take H the double of D, K its triple, and so on, until the multiple of D be that which first exceeds FG. Let this multiple be L; and let K be the next preceding one.

Since EF is the same multiple of AC that FG is of CB, whatever multiple FG is of CB, the same multiple (v. 1) are EG and FG of AC and CB; that is, EG, FG are equimultiples of AB, CB. And because L is supposed to be the multiple of D which first exceeds FG, the next preceding multiple K of D cannot exceed FG, and hence FG is not less than K; but by constn EF is greater than D: therefore the whole EG is greater than K and D. But since K is the multiple of D next preceding L, K and D are equal to L: therefore EG is greater than L, and we know that FG is not greater than L. Now since there are four magnitudes AB, D, BC, D, and there have been taken of the first AB and third BC certain equimultiples EG, FG, and of the second and fourth D a certain multiple L, such that the multiple of the first EG is greater than that of the second L, while the multiple of the third FG is not greater than that of the fourth L: therefore by the defn of one ratio being

greater than another (v. Def. 7), AB has to D a greater ratio than BC has to D. Which was to be proved.

II. D shall have a greater ratio to BC than it has to AB.

Construct as in Part I. of the prop^{n}.

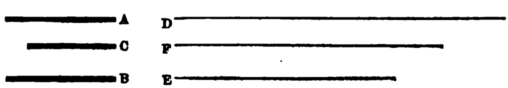
Then it may be shewn as before that FG, EG are equimultiples of BC, AB; and that L is greater than FG, but not greater than EG. Now since there are four magnitudes D, BC, D, AB, and there have been taken of the first and third D a certain multiple L, and of the second BC and fourth AB certain equimultiples FG, EG, such that the multiple of the first L is greater than that of the second FG, while the multiple of the third L is not greater than that of the fourth EG: therefore by the def^{n} of one ratio being greater than another, D has to CB a greater ratio than D has to AB. Which was to be proved.

PROP. IX. THEOR.

Two magnitudes which have each of them the same ratio to a third magnitude shall be equal to one another. And two magnitudes to each of which a third magnitude has the same ratio shall be equal to one another.

I. Let the two magnitudes A, B have each of them the same ratio to a third magnitude C; that is, let A be to C as B is to C. Then A shall be equal to B.

For if not: let them, if possible, be unequal, and let A be the one which is greater than the other B. Then of the two unequal magnitudes A and B, the greater A has a greater (v. 8) ratio to the third C than the less B has; and therefore by the def^{n} of one ratio being greater than another (v. Def. 7), there can be taken some equimultiples of A and B and some multiple of C such that the multiple of A is greater than that of C, but the multiple of B is not greater than that of C. Let such multiples be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, so that D is greater than F, but E is not greater than F. Now since by hyp^{s} A is to C as B to



C; and there have been taken of the first A and third B the equimultiples D, E, and of the second and fourth C the multiple F: therefore by the def^n of proportion (V. Def. 5), E is greater than, equal to, or less than F, according as D is greater than, equal to, or less than F. And D is greater than F; hence also E is greater than F. But E is not greater than F: which is impossible. Therefore A and B are not unequal; that is, A is equal to B. Which was to be proved.

II. To each of the two magnitudes A and B let a third magnitude C have the same ratio; that is, let C be to A as C is to B. Then A shall be equal to B.

For if not: let them, if possible, be unequal, and let A be the one which is greater than the other B. Then of the two unequal magnitudes A and B, the third magnitude C has a greater (V. 8) ratio to the less B than it has to the greater A; and therefore by the def^n of one ratio being greater than another (V. Def. 7), there can be taken some multiple of the first and third C, and some equimultiples of the second B and fourth A, such that the multiple of C is greater than that of B, but the multiple of C is not greater than that of A. Let such multiples be taken; and let F be the multiple of C, and E, D the equimultiples of B, A, so that F is greater than E, but F is not greater than D. Now since by hyp^s C is to A as C to B; and there have been taken of the first and third C the multiple F, and of the second A and fourth B the equimultiples D, E: therefore by the def^n of proportion (Def. 5) F is greater than, equal to, or less than E, according as F is greater than, equal to, or less than D. And F is not greater than D; hence F is not greater than E. But F is greater than E: which is impossible. Therefore A and B are not unequal; that is, A is equal to B. Which was to be proved.

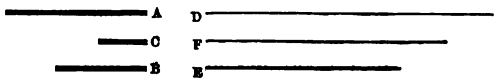
PROP. X. THEOR.

Of two magnitudes

(1) the one which has a greater ratio to a third magnitude than the other has shall be the greater;

(2) the one to which a third magnitude has a greater ratio than it has to the other shall be the less.

I. Let A, B be two magnitudes, and let A have to a third magnitude C a greater ratio than B has to C. Then A shall be greater than B.



By hyp^{s} A has a greater ratio to C than B has to C; and therefore by the def^{n} of one ratio being greater than another (v. Def. 7), there can be taken some equimultiples of the first A and third B, and some multiple of the second and fourth C, such that the multiple of A is greater than that of C, but the multiple of B is not greater than that of C. Let such multiples be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, so that D is greater than F, but E is not greater than F. Therefore D is greater than E; but D, E are equimultiples of A, B, and of two magnitudes the first is greater than the second, if the multiple of the first is greater than the same multiple of the second (v. Ax. 4): therefore A is greater than B. Which was to be proved.

II. Let A, B be two magnitudes; and let a third magnitude C have a greater ratio to B than it has to A. Then B shall be less than A.

By hyp^{s} C has a greater ratio to B than C has to A; and therefore by the def^{n} of one ratio being greater than another (v. Def. 7), there can be taken some multiple of the first and third C, and some equimultiples of the second B and fourth A, such that the multiple of C is greater than that of B, but the multiple of C is not greater than that of A. Let such multiples be taken; and let F be the multiple of C, and E, D the equimultiples of BA, so that F is greater than E, but F is not greater than D. Therefore E is less than D; but E, D are equimultiples of B, A, and of two magnitudes the first is less than the second, if the multiple of the first be less than

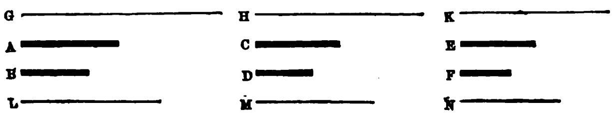
the same multiple of the second (v. Ax. 4): therefore B is less than A. Which was to be proved.

PROP. XI. THEOR.

Ratios that are the same to the same ratio shall be the same to one another.

Let the ratio which A has to B be the same as the ratio which C has to D, and the ratio which C has to D be the same as the ratio which E has to F; i. e. let A be to B as C to D, and C be to D as E to F. Then the ratio which A has to B shall be the same as the ratio which E has to F; i. e. A shall be to B as E to F.

Of A, C, E take any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N.



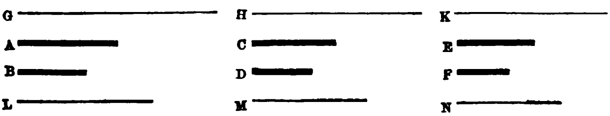
Then since A is to B as C to D by hyp^s, and there have been taken of the first A and third C the equimultiples G, H, and of the second B and fourth D the equimultiples L, M: therefore by the def^n of proportion (v. Def. 5) H is greater than, equal to, or less than M according as G is greater than, equal to, or less than L. In like manner it may be shewn, since C is to D as E to F by hyp^s, that K is greater than, equal to, or less than N, according as H is greater than, equal to, or less than M. And H has been shewn to be greater than, equal to, or less than M, according as G is greater than, equal to, or less than L: hence also K is greater than, equal to, or less than N, according as G is greater than, equal to, or less than L. Now since there are four magnitudes A, B, E, F, and there have been taken of the first A and third E any equimultiples whatever G, K, and of the second B and fourth F any equimultiples whatever L, N; and since it has been shewn that the multiple of the third K is greater than, equal to, or less than that of the fourth N, according as the multiple

of the first G, is greater than, equal to, or less than that of the second L: therefore by the def° of proportion A is to B as E to F. Which was to be proved.

PROP. XII. THEOR.

If any number of ratios be the same to one another: then any one of the antecedents shall be to its consequent as all the antecedents taken together are to all the consequents taken together.

Let any number of ratios, viz. the ratio of A to B, the ratio of C to D, the ratio of E to F, be the same to one another. Then any one of the antecedents A shall be to its consequent B as all the antecedents A, C, E together are to all the consequents B, D, F together.



Of A, C, E take any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N.

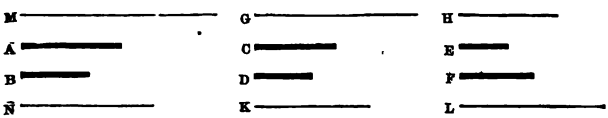
Since A is to B as C to D by hyp[s], and there have been taken of the first A and third C any equimultiples whatever G, H, and of the second B and fourth D any equimultiples whatever L, M: therefore by the def[n] of proportion (v. Def. 5), H is greater than, equal to, or less than M, according as G is greater than, equal to, or less than L. In like manner it may be shewn, since A is to B as E to F by hyp[s], that K is greater than, equal to, or less than N, according as G is greater than, equal to, or less than L. Hence G, H, K together are greater than, equal to, or less than L, M, N together according as G is greater than, equal to, or less than L. And because G, H, K are by const[n] equimultiples of A, C, E, whatever multiple G is of A, the same are (v. 1) all G, H, K together of A, C, E together; that is, G and G, H, K together are equimultiples of A, and A, C, E together: in like manner, L and L, M, N together are equimultiples of B, and B, D, F together. Now since there are four magnitudes A, B, all A, C, E together, and all B,

D, F together, and there have been taken of the first A and the third A, C, E together any equimultiples whatever G and G, H, K together, and of the second B and fourth B, D, F together any equimultiples whatever L and L, M, N together; and since it has been shewn that the multiple of the third G, H, K together is greater than, equal to, or less than that of the fourth L, M, N together, according as the multiple of the first G is greater than, equal to, or less than that of the second L: therefore by the def^{n} of proportion A is to B as all A, C, E together are to all B, D, F together. Which was to be proved.

PROP. XIII. THEOR.

If the first magnitude have to the second the same ratio which the third has to the fourth, and the third have to the fourth a greater ratio than the fifth has to the sixth: then the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A have the same ratio to B which C has to D, and let C have to D a greater ratio than E has to F. Then also shall A have to B a greater ratio than E has to F.



By hyp^{s} C has a greater ratio to D than E has to F; and therefore by the def^{n} (V. Def. 7) of one ratio being greater than another, there can be taken some equimultiples of the first C and the third E and some equimultiples of the second D and fourth F, such that the multiple of C is greater than that of D, but the multiple of E is not greater than that of F. Let such be taken; and let G, H be the equimultiples of C, E and K, L the equimultiples of D, F, so that G is greater than K, but H is not greater than L. Also whatever multiple G is of C, of A take the same multiple M; and whatever multiple K is of D, take N the same multiple of B.

Then since A is to B as C to D by hyp[s], and there have been taken of the first A and third C the equimultiples M, G, and of the second B and fourth D the equimultiples N, K: therefore by the def[n] of proportion (v. Def. 5) G is greater than, equal to, or less than K, according as M is greater than, equal to, or less than N. But G is greater than K: hence likewise M must be greater than N, for had M been either equal to or less than N, G must have been either equal to or less than K. Also H is not greater than L. Now, since there are four magnitudes A, B, E, F, and there have been taken of the first A and third E some equimultiples M, H, and of the second B and fourth F some equimultiples N, L, such that the multiple of the first M is greater than that of the second N, while the multiple of the third H is not greater than that of the fourth L: therefore by the def[n] of one ratio being greater than another, A has a greater ratio to B than E has to F. Which was to be proved.

COR.—If the first magnitude have to the second the same ratio which the third has to the fourth, and the fifth have to the sixth a greater ratio than the first have to the second: then the fifth shall also have to the sixth a greater ratio than the third has to the fourth.

The proof is similar to that of the prop[n].

PROP. XIV. THEOR.

If four magnitudes, which are all of the same kind, be proportional: then the second shall be greater than, equal to, or less than the fourth, according as the first is greater than, equal to, or less than the third.

Let the four magnitudes A, B, C, D, which are all of the same kind, be proportional, i.e. let A be to B as C is to D. Then the second B shall be greater than, equal to, or less than the fourth D, according as the first A is greater than, equal to, or less than the third C.

There are three cases, according as A is greater than, equal to, or less than C.

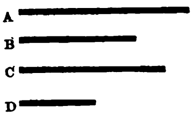


Fig. 1.

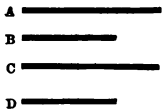


Fig. 2.

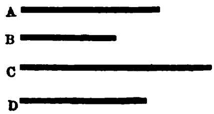


Fig. 3.

I. Let A be greater than C (Fig. 1).

By hyp^{s} A is to B as C to D, or C is to D as A to B (v. Def. 5, Obs.); and of the two unequal magnitudes A and C, the greater A has a greater (v. 8) ratio to a third B of the same kind than C has to B; therefore C has a greater (v. 13) ratio to D than C has to B. But of two magnitudes that to which a third has a greater ratio than it has to the other, is the less (v. 10); hence D is less than B, that is, B is greater than D.

II. Let A be equal to C (Fig. 2).

Since A is supposed equal to C, and A is to B as C to D, C is to B as C to D. But two magnitudes to each of which a third has the same ratio are equal (v. 9); therefore B is equal to D.

III. Let A be less than C (Fig. 3).

Since C is to D as A to B, and the first C is supposed greater than A: therefore by Case I. of the prop^{n}, the second D is likewise greater than the fourth B; that is, B is less than D.

Hence B is greater than, equal to, or less than D, according as A is greater than, equal to, or less than C. Which was to be proved.

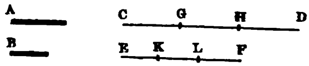
PROP. XV. THEOR.

Magnitudes shall have the same ratio to one another which their equimultiples have; i. e. one magnitude shall have the same ratio to another magnitude of the same kind which any multiple of the first has to the same multiple of the second.

Let A, B be two magnitudes of the same kind; CD, EF equimultiples of them respectively. Then shall A be to B as CD is to EF.

By hyp^{s} CD, EF are equimultiples of A, B; and hence

CD can be divided into as many magnitudes, each equal to A, as EF can into magnitudes, each equal to B. Let then CD be divided into magnitudes CG, GH, HD, each equal to A, and EF into magnitudes EK, KL, LF, each equal to B; the number of CG, GH, HD being equal to that of the others EK, KL, LF.



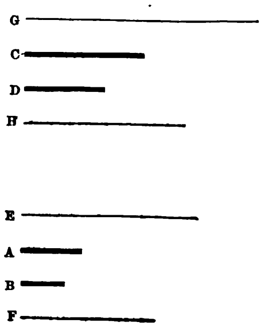
Then since by constⁿ CG, GH, HD are all equal, and likewise EK, KL, LF are all equal, the ratio which CG has to EK, the ratio which GH has to KL, and the ratio which HD has to LF are all the same to one another (V. 7): therefore one of the antecedents CG is to its consequent EK as all the antecedents CG, GH, HD together are to all the consequents EK, KL, LF together (V. 12); that is, CG is to EK as CD to EF. Now by constⁿ CG is equal to A, and EK to B; therefore A is to B as CD to EF. Which was to be proved.

PROP. XVI. THEOR.

If four magnitudes, which are all of the same kind, be proportional: then they shall also, when taken alternately, be proportional; i. e. the first shall be to the third as the second is to the fourth.

OBS. This propⁿ is usually cited by the word "alternando," or "permutando."

Let A, B, C, D be four magnitudes of the same kind, and let A be to B as C is to D. Then shall A be to C as B is to D.



Of A, B take any equimultiples whatever E, F; and of C, D any equimultiples whatever G, H.

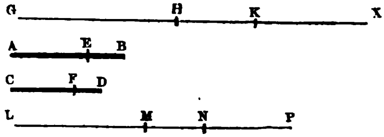
Then since by constⁿ E, F are equimultiples of A, B, and magnitudes have the same ratio to one another which their equimultiples have (V. 15), A is to B as E to F. In

like manner it may be shewn that C is to D as G to H. But by hyp[e] A is to B as C to D; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore E is to F as G to H. But when four magnitudes are proportionals, the second is greater than, equal to, or less than the fourth, according as the first is greater than, equal to, or less than the third (V. 14): hence F is greater than, equal to, or less than H, according as E is greater than, equal to, or less than G. Now, since there are four magnitudes A, C, B, D, and of the first A and third B are taken any equimultiples whatever E, F, and of the second C and fourth D any equimultiples whatever G, H; and since it has been shewn that the multiple of the third F is greater than, equal to, or less than that of the fourth H, according as the multiple of the first E is greater than, equal to, or less than that of the second G: therefore by the def[n] of proportion (V. Def. 5) A is to C as B to D. Which was to be proved.

PROP. XVII. THEOR.

If four magnitudes be proportional, the first being greater than the second, and the third than the fourth: then the excess of the first above the second shall be to the second as the excess of the third above the fourth is to the fourth.

OBS. This prop[n] is usually cited by the word "dividendo."



Let AB be to BE as CD is to DF, AB being greater than BE, and consequently (V. A) CD than DF. Then shall AE, the excess of AB above BE, be to BE as CF, the excess of CD above DF, is to DF.

Of AE, EB, CF, FD take any equimultiples whatever GH, HK, LM, MN; and again of EB, FD take any equimultiples whatever KX, NP.

Since by const[n] GH, HK are equimultiples of AE, BE

whatever multiple GH is of AE, the same multiple is (V. 1) GK of AB. For like reason whatever multiple LM is of CF, the same multiple is LN of CD; but by const^n GH, LM are equimultiples of AE, CF: therefore also GK, LN are equimultiples of AB, CD. Again, since by const^n HK, MN are equimultiples of EB, FD; and KX, NP are also equimultiples of EB, FD: therefore HK together with KX, that is, HX, and MN together with NP, that is, MP, are equimultiples (V. 2) of EB, FD. And because AB is to BE as CD to FD by hyp^s, and there have been taken of the first AB and third CD the equimultiples GK, LN, and of the second BE and fourth FD the equimultiples HX, MP: therefore by the def^n of proportion (V. Def. 5) LN is greater than, equal to, or less than MP, according as GK is greater than, equal to, or less than HX. But, taking away the common part MN from each of LN and MP, LM is greater than, equal to, or less than (Ax. 3. Ax. 5) NP, according as LN is greater than, equal to, or less than MP; and taking away the common part HK from each of GK and HX, GH is greater than, equal to, or less than KX, according as GK is greater than, equal to, or less than HX: hence LM is greater than, equal to, or less than NP, according as GH is greater than, equal to, or less than KX. Now, since there are four magnitudes AE, EB, CF, FD, and there have been taken of the first AE and third CF any equimultiples whatever GH and LM, and of the second EB and fourth FD any equimultiples whatever KX and NP; and since it has been shewn that the multiple of the third LM is greater than, equal to, or less than that of the fourth MP, according as the multiple of the first GH is greater than, equal to, or less than that of the second KX: therefore by the def^n of proportion AE is to EB as CF to FD. Which was to be proved.

PROP. XVIII. THEOR

If four magnitudes be proportional: then the first together with the second shall be to the second as the third together with the fourth is to the fourth.

OBS. This prop^n is usually cited by the word "componendo."

Let AE be to EB as CF is to FD. Then shall AE toge-

ther with EB, that is, AB be to EB, as CF together with FD, that is, CD is to FD.

Of AB, BE, CD, DF take any equimultiples whatever GH, HK, LM, MN; and again of BE, DF take any equimultiples whatever KO, NP.

Then since by const^{n} KO, NP are equimultiples of BE, DF, and KH, NM likewise are equimultiples of BE, DF, NP will be greater than, equal to, or less than NM, according as KO is greater than, equal to, or less than KH.

I. Let KO be either equal to KH, and accordingly NP equal to NM (Fig. 1), or less than KH, and accordingly NP less than NM (Fig. 2).

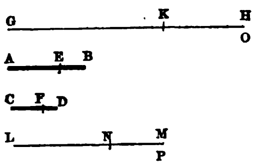


Fig. 1.

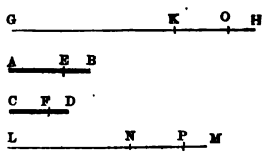


Fig. 2.

By const^{n} GH, HK are equimultiples of AB, BE, of which AB is the greater; and the multiple of a greater magnitude is greater than the same multiple of a less (V. Ax. 3): therefore GH is greater than HK. But in neither figure is KO greater than HK; therefore GH is greater than KO, and it may be shewn in like manner that LM is greater than NP.

II. Let KO be greater than KH, and accordingly NP greater than NM (Fig^{s} 3 and 4).

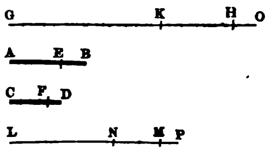


Fig. 3.

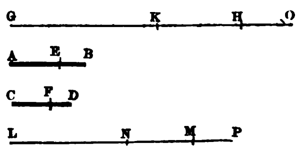


Fig. 4.

Since GH, HK are equimultiples of AB, BE, whatever multiple GH is of AB, the same multiple (V. 5) is GK, the

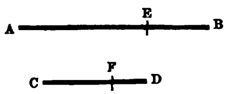
excess of GH above HK, of AE, the excess of AB above BE. For like reason whatever multiple LM is of CD, the same multiple is LN of CF; but GH, LM are by const^n equimultiples of AB, CD: therefore GK, LN are equimultiples of AE, CF. Again, if from KO, NP, which are equimultiples of BE, DF, there be taken KH, NM, which are also equimultiples of BE, DF, the remainders HO, MP are (v. 6) either equal to BE, DF (Fig. 3), or else equimultiples of BE, DF (Fig. 4). In Fig. 3, because AE is to EB as CF to FD by hyp^s, and there have been taken of the first AE and third CF equimultiples GK, LN, therefore (v. 4, Cor.) GK is to EB as LN to FD; but HO is supposed equal to EB, and MP to FD: therefore GK is to HO as LN to MP, and hence LN is greater than, equal to, or less than (v. A) MP, according as GK is greater than, equal to, or less than HO. In Fig. 4, because AE is to EB as CF to FD, and there have been taken of the first AE and third CF equimultiples GK, LN, and of the second EB and fourth FD equimultiples HO, MP: therefore by the def^n of proportion (v. Def. 5) LN is greater than, equal to, or less than MP, according as GK is greater than, equal to, or less than HO. But in both figures, adding NM to each of LN and MP, and HK to each of GK and HO, LM is greater than, equal to, or less than (Ax^s 2 and 4) NP, according as LN is greater than, equal to, or less than MP, and GH is greater than, equal to, or less than KO, according as GK is greater than, equal to, or less than HO: therefore in Case II. LM is greater than, equal to, or less than NP, according as GH is greater than, equal to, or less than KO.

Also in Case I. it was shewn that LM was greater than NP, and at the same time GH greater than KO. Now since in both Cases I. and II. there are four magnitudes AB, BE, CD, DF, and there have been taken of the first AB and third CD any equimultiples whatever GH, LM, and of the second BE and fourth DF any equimultiples whatever KO, NP; and since it has been shewn that the multiple of the third LM is in all cases greater than, equal to, or less than that of the fourth NP, according as GH, the multiple of the first, is greater than, equal to, or less than that of the second KO: therefore by the def^n of proportion, AB, is to BE as CD to DF. Which was to be proved.

PROP. XIX. THEOR.

If four magnitudes, which are all of the same kind, be proportional, the first being greater than the third, and the second than the fourth: then the excess of the first above the third, shall be to that of the second above the fourth as the first is to the second.

Let the four magnitudes AB, CD, AE, CF, which are all of the same kind, be proportional; AB being greater than AE, and consequently (v. 14) CD than CF. Then EB, the excess of AB above AE, shall be to FD, the excess of CD above CF, as AB is to CD.



By hyp^s AB is to CD as AE to CF; and the four magnitudes are of the same kind: therefore, alternando (v. 16), AB is to AE as CD to CF. Now by hyp^s AB is greater than AE, and CD than CF: therefore, dividendo (v. 17), BE is to AE as DF to CF, and hence, alternando, BE is to DF as AE to CF. But AB is to CD as AE to CF; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore BE is to DF as AB to CD. Which was to be proved.

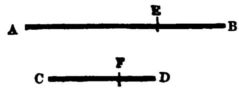
COR.—If four magnitudes, which are all of the same kind, be proportional, the first being greater than the third, and the second than the fourth: then the excess of the first above the third shall be to that of the second above the fourth as the third is to the fourth.

For it was shewn in the proof of the prop^n that BE is to DF as AE to CF. Which was to be proved.

PROP. E. THEOR.

If four magnitudes be proportional, the first being greater than the second, and the third than the fourth: then the first shall be to its excess above the second, as the third is to its excess above the fourth.

Obs. This prop^n is usually cited by the word "convertendo."



Let AB be to BE as CD is to DF; AB being greater than BE, and consequently (v. A) CD than DF. Then AB shall be to AE, its excess above BE, as CD is to CF, its excess above DF.

By hyp^s AB is to BE as CD to DF, AB being greater than BE and CD than DF; therefore, dividendo (v. 17), AE is to BE as CF to FD: therefore, invertendo (v. B), BE is to AE as FD to CF. Wherefore, componendo (v. 18), BE together with AE is to AE as FD together with CF is to CF; that is, BA is to AE as DC to CF. Which was to be proved.

PROP. XX. THEOR.

If the first magnitude be to the second as the third is to the fourth, and if the second be to the fifth as the fourth is to the sixth: then the third shall be greater than, equal to, or less than the sixth, according as the first is greater than, equal to, or less than the fifth.

Let A be to B as C is to D, and let B be to E as D is to F. Then C shall be greater than, equal to, or less than F, according as A is greater than, equal to, or less than E.

There are three cases according as A is greater than, equal to, or less than E.

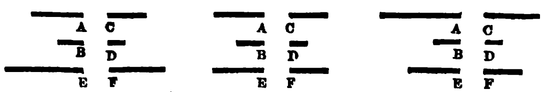


Fig. 1. Fig. 2. Fig. 3.

I. Let A be greater than E (Fig. 1).

By hyp^s C is to D as A to B, and of the two unequal magnitudes A and E, the greater A has a greater (v. 8) ratio to a third B of the same kind than the less E has to B: therefore C has to D a greater ratio (v. 13) than E has to B. Now by hyp^s B is to E as D to F; therefore, invertendo (v. B), E is to B as F to D; and C was shewn to have to D a greater ratio than E has to B: therefore C

has to D a greater ratio (V. 13, Cor.) than F has to D. But of two magnitudes the one which has a greater ratio to a third magnitude than the other has to it is the greater (V. 10): therefore C is greater than F.

II. Let A be equal to E (Fig. 2).

Since A, E are supposed equal; and two equal magnitudes have each of them the same ratio to a third of the same kind (V. 7): therefore A is to B as E to B. But A is to B as C to D; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore C is to D as E to B. Now B is to E as D to F; therefore, invertendo, E is to B as F to D. And it was shewn that C is to D as E to B, and ratios that are the same to the same ratio are the same to one another; therefore C is to D as F to D. But two magnitudes which have each of them the same ratio to a third, are equal (V. 9): therefore C is equal to F.

III. Let A be less than E (Fig. 3).

Because A is to B as C to D, and B is to E as D to F; therefore, invertendo, B is to A as D to C, and E is to B as F to D. Also since A is supposed less than E, E is greater than A. Hence E is to B as F to D, and B is to A as D to C, and the first E is greater than the fifth A; therefore by Case I. the third F is greater than the sixth C, that is, C is less than F.

Hence it has been shewn that C is greater than, equal to, or less than F, according as A is greater than, equal to, or less than E. Which was to be proved.

PROP. XXI. THEOR.

If the first magnitude be to the second as the third is to the fourth, and if the second be to the fifth as the sixth is to the third: then the sixth shall be greater than, equal to, or less than the fourth, according as the first is greater than, equal to, or less than the fifth.

Let A be to B as C is to D, and let B be to E as F is to C. Then F shall be greater than, equal to, or less than D, according as A is greater than, equal to, or less than E.

There are three cases according as A is greater than, equal to, or less than E.

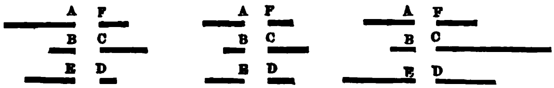


Fig. 1. Fig. 2. Fig. 3.

I. Let A be greater than E (Fig. 1).

By hyp^s C is to D as A to B; and of the two unequal magnitudes A and E, the greater A has a greater ratio (v. 8) to a third B of the same kind than the less E has to B: therefore C has to D a greater ratio (v. 13) than E has to B. Now by hyp^s B is to E as F to C, and therefore, invertendo (v. B), E is to B as C to F; and C was shewn to have to D a greater ratio than E has to B: therefore C has to D a greater ratio (v. 13, Cor.) than C has to F. But of two magnitudes that to which a third has a greater ratio than it has to the other is the less (v. 10): therefore D is less than F, that is, F is greater than D.

II. Let A be equal to E (Fig. 2).

Since A, E are supposed equal; and two equal magnitudes have each of them the same ratio to a third of the same kind (v. 7): therefore A is to B as E to B. But A is to B as C to D; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore E is to B as C to D. Now B is to E as F to C; therefore, invertendo, E is to B as C to F. And it was shewn that E is to B as C to D; and ratios that are the same to the same ratio are the same to one another: therefore C is to F as C to D. But two magnitudes to each of which a third has the same ratio are equals (v. 9): therefore F is equal to D.

III. Let A be less than E (Fig. 3).

Because A is to B as C to D, and B is to E as F to C; therefore, invertendo, B is to A as D to C, and E is to B as C to F. Also since A is supposed less than E, E is greater than A. Hence E is to B as C to F, and B is to A as D to C, and the first E is greater than the fifth A; therefore by Case I. the sixth D is greater than the fourth F, that is, F is less than D.

Hence it has been shewn that F is greater than, equal to, or less than D, according as A is greater than, equal to, or less than E. Which was to be proved.

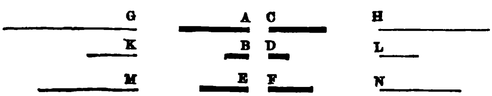
PROP. XXII. THEOR.

If the first magnitude be to the second as the third is to the fourth, and if the second magnitude be to the fifth as the fourth is to the sixth: then the first shall be to the fifth as the third is to the sixth.

OBS. This prop^n is usually cited by the words "ex æquali" or "ex æquo."

Let A be to B as C is to D, and let B be to E as D is to F. Then shall A be to E as C is to F.

Of A, C take any equimultiples whatever G, H; of B, D any equimultiples whatever K, L; and of E, F any equimultiples whatever M, N.



By hyp^s A is to B as C to D; and there have been taken of the first A and third C equimultiples G, H, and of the second B and fourth D equimultiples K, L: therefore (V. 4) G is to K as H to L. For like reason K is to M as L to N. Therefore, since G is to K as H to L, and K is to M as L to N, H is greater than, equal to, or less than (V. 20) N, according as G is greater than, equal to, or less than M. Now since there are four magnitudes A, E, C, F, and of the first A and third C there have been taken any equimultiples whatever G, H, and of the second E and fourth F any equimultiples whatever M, N; and since it has been shewn that the multiple of the third H is greater than, equal to, or less than that of the fourth N according as the multiple of the first G is greater than, equal to, or less than that of the second M: therefore by the def^n of proportion (V. Def. 5) A is to E as C to F. Which was to be proved.

COR.—If there be any number of proportions so constituted that the second and fourth terms of each form respectively the first and third terms of the next: then the first term of the first proportion shall be to the second of the last as the third of the first proportion is to the fourth of the last.

For let the number of proportions so constituted be three; and let A be to B as C to D, B be to E as D to F, and E to G as F to H. Then shall A be to G as C to H.

$$A : \cancel{B} :: C : \cancel{D}$$
$$\cancel{B} : \cancel{E} :: \cancel{D} : \cancel{F}$$
$$\cancel{E} : G :: \cancel{F} : H$$
$$A : G :: C : H$$

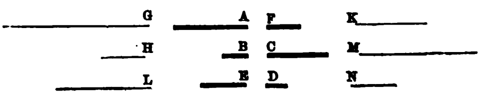
By the propⁿ, since A is to B as C to D, and B is to E as D to F, therefore A is to E as C to F. And again by the propⁿ, since A is to E as C to F, and E to G as F to H, therefore A is to G as C to H. And the same might be shewn, if there were four or more proportions. Which was to be proved.

PROP. XXIII. THEOR.

If the first magnitude be to the second as the third is to the fourth, and if the second be to the fifth as the sixth is to the third; then the first shall be to the fifth as the sixth is to the fourth.

OBS.—This propⁿ is usually cited by the words "ex æquali in proportione perturbatâ," or "ex æquo perturbato."

Let A be to B as C is to D, and let B be to E as F is to C. Then shall A be to E as F is to D.



Of A, B, F take any equimultiples whatever G, H, K; and of E, C, D any equimultiples whatever L, M, N.

By constⁿ G, H are equimultiples of A, B; and magnitudes have the same ratio to one another that their equi-

multiples have (V. 15): therefore A is to B as G to H. For like reason C is to D as M to N. But by hyp^{s} A is to B as C to D; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore G is to H as M to N. Again by hyp^{s} B is to E as F to C; and there have been taken of the first B and third F equimultiples H, K, and of the second E and fourth C equimultiples L, M; therefore (V. 15) H is to L as K to M. But it has been shewn that G is to H as M to N, and H is to L as K to M: therefore the sixth K is greater than, equal to, or less than (V. 21) the fourth N, according as the first G is greater than, equal to, or less than the fifth L. Now since there are four magnitudes A, E, F, D, and there have been taken of the first A and third F any equimultiples whatever G, K, and of the second E and fourth D any equimultiples whatever L, N; and since it has been shewn that the multiple of the third K is greater than, equal to, or less than that of the fourth N, according as the multiple of the first G is greater than, equal to, or less than that of the second L: therefore by the def^{n} of proportion (V. Def. 5) A is to E as F to D. Which was to be proved.

COR.—If there be any number of proportions so constituted that the second and third terms of each form respectively the first and fourth terms of the next: then the first term of the first proportion shall be to the second of the last, as the third term of the last proportion is to the fourth of the first.

For let the number of proportions so constituted be three; and let A be to B as C to D, B be to E as F to C, and E be to G as H to F. Then A shall be to G as H is to D.

A	:	B	::	C	:	D
B	:	E	::	F	:	C
E	:	G	::	H	:	F
A	:	G	::	H	:	D

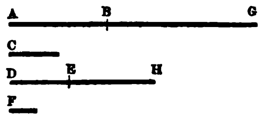
By the prop^{n} since A is to B as C to D, and B is to E as F to C, therefore A is to E as F to D. And again by the prop^{n}, since A is to E as F to D, and E is to G as H to F, therefore A is to G as H is

to D. And the same might be shewn if there were four or more proportions. Which was to be proved.

PROP. XXIV. THEOR.

If the first magnitude be to the second as the third is to the fourth, and if the fifth be to the second as the sixth is to the fourth: then the first and fifth together shall be to the second as the third and sixth together are to the fourth.

Let AB be to C as DE is to F, and let BG be to C as EH is to F. Then AB and BG together, that is, AG shall be to C as DE and EH together, that is, DH is to F.

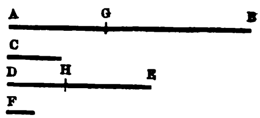


By hyp[s] BG is to C as EH to F; therefore, invertendo (v. B), C is to BG as F to EH. Again by hyp[s] AB is to C as DE to F; and it has been just shewn that C is to BG as F to EH: therefore, ex æquali (v. 22), AB is to BG as DE to EH. Hence, componendo (v. 18), AG is to BG as DH to EH; and BG is to C as EH to F: therefore, ex æquali, AG is to C as DH to F. Which was to be proved.

COR. 1.—If the first magnitude be to the second as the third is to the fourth, and if the fifth be to the second as the sixth is to the fourth, the first being greater than the fifth and the third than the sixth: then the excess of the first above the fifth shall be to the second as that of the third above the sixth is to the fourth.

Let AB be to C as DE is to F, and let BG be to C as EH to F; AB being greater than BG, and therefore, since it may be shewn as in the prop[n] that AB is to BG as DE to EH, DE likewise greater (v. A) than EH. Then AG, the excess of AB above BG, shall be to C as DH, the excess of DE above EH, is to F.

Since AB is to BG as DE to EH, AB being greater



than BG and consequently DE than EH; therefore, dividendo (v. 17), AG is to BG as DH to EH; and BG is to C as EH to F: therefore, ex æquali, AG is to C as DH to F. Which was to be proved.

COR. 2.—If there be any number of proportions so constituted that the second term of each is the same throughout, and likewise the fourth: then all the first terms together shall be to the common second term as all the third terms together are to the common fourth term.

For let the number of proportions so constituted be three; and let A be to B as C to D, E be to B as F to D, and G be to B as H to D. Then A, E, G together shall be to B as C, F, H together are to D.

A	:	B	::	C	:	D
E	:	B	::	F	:	D
G	:	B	::	H	:	D

By the propⁿ, since A is to B as C to D, and E is to B as F to D, therefore A and E together are to B as C and F together are to D. And again by the propⁿ, since A, E together are to B as C, F together to D, and G is to B as H to D, therefore A, E, G together are to B as C, F, H together are to D. And the same might be shewn if there were four or more proportions. Which was to be proved.

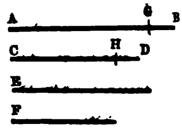
PROP. XXV. THEOR.

If four magnitudes which are all of the same kind be proportional, the first being the greatest: then the first and fourth together shall be greater than the second and third together.

Let AB be to CD as E to F, the four magnitudes being all of the same kind and AB the greatest. Then AB

together with F shall be greater than CD together with E.

By hyp^s AB is to CD as E to F, and AB is greater than CD; therefore also E is greater than (v. A) F. Again by hyp^s AB, CD, E, F are all of the same kind, and AB is greater than E; therefore also CD is greater than (v. 14) F. Take then the magnitude AG equal to E, and CH equal to F.



Since AB is to CD as E to F, and by const^n AG is equal to E, and CH to F, AB is to CD as AG to CH. But because AB is to CD as AG to CH, the four magnitudes being all of the same kind, and AB, CD respectively greater than AG, CH; therefore BG, the excess of AB above AG, is (v. 19) to DH, the excess of CD above CH, as AB to CD, or AB is to CD as BG to DH. Now AB is greater than CD; therefore also BG is greater than DH. Again since AG is equal to E, and F to CH; therefore, adding equals to equals, AG together with F is equal to (Ax. 2) CH together with E. To the unequal magnitudes BG, HD of which GB is the greater, add the equals AG, F together and CH, E together respectively, therefore BG, AG, F together are greater than (Ax. 4) HD, CH, E together, that is, AB and F together are greater than CD and E together. Which was to be proved.

THE

ELEMENTS OF EUCLID.

BOOK VI.

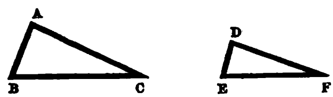
DEFINITIONS.

I.

One polygon is defined to be similar to another polygon having the same number of sides, when each angle of the one is equal to an angle of the other, and the sides about each pair of equal angles are proportional; that is, one of the sides including each angle in one polygon is to the other including that angle as one of the sides including the angle in the other polygon equal to it is to the other side including that angle.

Obs. 1. For two polygons to be similar to one another, there are twice as many conditions to be satisfied as either polygon has sides. This will appear more clearly from the following examples:—

Ex. 1. In order for the two triangles ABC, DEF to be similar, we must have



(1) the angle ABC equal to the angle DEF;

(2) the angle ACB equal to the angle DFE;

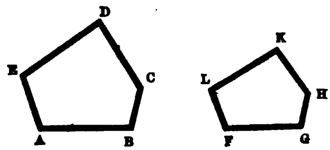
(3) the angle BAC equal to the angle EDF;

(4) the sides about the pair of equal angles ABC, DEF proportional, viz. AB to BC as DE to EF;

(5) the sides about the pair of equal angles ACB, DFE proportional, viz. BC to CA as EF to FD;

(6) the sides about the pair of equal angles BAC, EDF proportional, viz. BA to AC as ED to DF.

Ex. 2. In order for the two pentagons ABCDE, FGHKL to be similar, we must have



(1) the angle A equal to the angle F;

(2) the angle B equal to the angle G;

(3) the angle C equal to the angle H;

(4) the angle D equal to the angle K;

(5) the angle E equal to the angle L;

(6) the sides about the pair of equal angles A, F proportional, viz. EA to AB as LF to FG;

(7) the sides about the pair of equal angles B, G proportional, viz. AB to BC as FG to GH;

(8) the sides about the pair of equal angles C, H proportional, viz. BC to CD as GH to HK;

(9) the sides about the pair of equal angles D, K proportional, viz. CD to DE as HK to KL;

(10) the sides about the pair of equal angles E, L proportional, viz. DE to EA as KL to LF.

Obs. 2. Each pair of sides in the polygons which form homologous terms (v. Def. 12) in the different proportions are called homologous sides.

Thus in Ex. 1 there are three pairs of homologous sides, viz. AB and DE; BC and EF; AC and DF. And in Ex. 2 there are five pairs of homologous sides, viz. AB and FG; BC and GH; CD and HK; DE and KL; EA and LF.

Obs. 3. It appears from this defn that regular polygons of the same number of sides are similar.

For if there be two regular polygons of the same number of

sides, all the angles of each together with four right angles is equal to twice as many right angles as the polygon has sides; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the sum of all the angles in each polygon together with four right angles, and therefore (Ax. 3) the sum alone, is the same. Hence the number of angles being alike in each, each angle of the one polygon is equal to each of the other; and the sides about the equal angles in each are equal, and have therefore to one another the same ratio, viz. a ratio of equality (v. Def. 4): therefore by the defn the two polygons are similar.

This observation will be referred to in the proof of Bk. xii. Prop. 2.

Obs. 4. It appears that polygons which are equal in every respect are similar.

For each angle of the one is equal to one angle of the other; and the ratio of the sides about each angle of the one is the same as that of the sides about the equal angle of the other, since the sides are equal respectively.

II.

Two triangles or parallelograms are said to be reciprocal to one another, when the sides about one pair of angles are proportional in such a manner, that one of the sides about the angle in one figure is to one of the sides about the angle in the other figure as the other about this angle is to the other about the angle in the first figure.

III.

A straight line is said to be cut in extreme and mean ratio, when it is divided into two parts such, that the whole line is to the greater part as the greater part is to the less.

IV.

The altitude of a polygon is a straight line drawn from one of its angular points perpendicular to one of the opposite sides, or that side produced.

Obs. 1. The angular point is called the vertex; and the opposite side is called the base.

Obs. 2. Since either of the angular points may be selected as the vertex, and again either of the opposite sides as the base, a polygon may have several different altitudes.

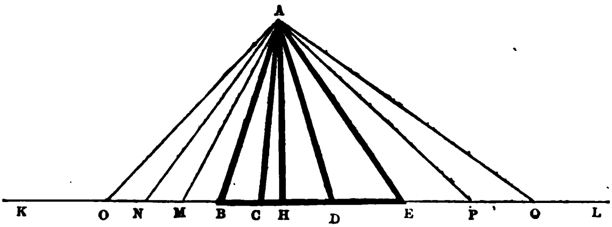
Thus in the case of a triangle, since each of the three angular points may be taken as the vertex, and to each vertex there is one opposite side for the base, a triangle can have three different altitudes, and no more. And in the case of a parallelogram, or any four-sided figure, since each of the four angular points may be taken as the vertex, and to each vertex there are two opposite sides for bases, a parallelogram or any four-sided figure can have eight different altitudes, and no more.

PROPOSITIONS.

PROP. I. THEOR.

Triangles or parallelograms, which have the same altitude, shall be to one another as their bases.

I. Let the two triangles ABC, ADE have the same altitude, viz. the straight line AH, drawn from the common vertex A perpendicular to BE, or BE produced, in which straight line are their bases BC, DE. Then the base BC shall be to the base DE as the triangle ABC is to the triangle ADE.



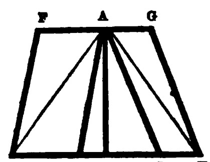
Produce BE both ways to K, L; from BK cut off (i. 3) any number of parts BM, MN, NO, each equal to BC; and from EL cut off any number of parts EP, PQ, each equal to DE. Join AM, AN, AO; AP, AQ.

Because the triangles ABC, AMB, ANM, AON are on equal bases BC, MB, NM, ON by constⁿ. and between the same parallels, they are all equal (i. 38); and hence whatever multiple CO is of BC the same multiple is the triangle AOC of the triangle ABC; that is, the straight line OC and the triangle AOC are equimultiples of the base BC and the triangle ABC. By like reasoning the straight line DQ and the triangle ADQ are equimultiples of the base DE and the

triangle ADE. Also if OC be equal to DQ, the triangles AOC, ADQ will be on equal bases OC, DQ, and between the same parallels, and therefore the triangle AOC will be equal to the triangle ADQ; if OC be greater than DQ, the triangle AOC will be greater than the triangle ADQ; and if less, less. Now, since there are four magnitudes, viz. the base BC, the base DE, the triangle ABC, the triangle ADE, and there have been taken of the first, the base BC, and the third, the triangle ABC, any equimultiples whatever, the straight line OC and the triangle ADE, and of the second, the base DE, and the fourth, the triangle ADE, any equimultiples whatever, the straight line DQ and the triangle ADQ; and since it has been shewn that the multiple of the third, the triangle AOC, is greater than, equal to, or less than that of the fourth, the triangle ADQ, according as the multiple of the first, the straight line OC, is greater than, equal to, or less than that of the second, the straight line DQ: therefore by the def^{n} of proportion (v. Def. 5), the base BC is to the base DE as the triangle ABC to the triangle ADE. Which was to be proved.

II. Let the two parallelograms AFBC, GADE have the same altitude, viz. the straight line AH drawn from the common vertex A perpendicular to BE, or BE produced, in which straight line are their bases BC, DE. Then the base BC shall be to the base DE as the parallelogram FC to the parallelogram GD.

Join AB, AE.



Because AB is a diagonal of the parallelogram FC, it bisects (i. 41) it, and FC is double of the triangle ABC. In like manner GD is double of the triangle ADE; and magnitudes have to one another the same ratio which their equimultiples have (v. 15): therefore the triangle ABC is to the triangle ADE as FC is to GD. But by Part I. of the prop^{n}, since the triangles ABC, ADE have the same altitude AH, BC is to DE as ABC to ADE; and ratios that are the same to the same ratio are the same to one

another (V. 11): therefore the base BC is to the base DE as the parallelogram FC to the parallelogram GD. Which was to be proved.

COR.—Triangles or parallelograms, which have equal altitudes, shall be to one another as their bases.

To prove this of two triangles, which have equal altitudes, let them be so placed that their vertices may coincide, and that the altitude of one may fall on that of the other: then since the altitudes are equal, they will coincide, and since the straight lines, in which their bases are, are each perpendicular to the altitudes, these straight lines will be in the same straight line. It may then be shewn by Part I. of the propⁿ, that these triangles, having the same altitude, are to one another as their bases; and what is proved of triangles may, as in Part II., be extended to parallelograms.

PROP. II. THEOR.

If a straight line drawn parallel to one of the sides of a triangle cut the other two sides, or these produced: then it shall cut them proportionally. And if two sides of a triangle, or these produced, be cut proportionally: then the straight line which joins the point of section shall be parallel to the third side of the triangle.

I. Let ABC be a triangle; and let DE be drawn parallel to one of its sides BC, cutting the other two AB, AC (Fig.

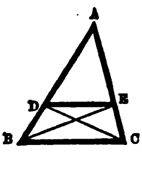


Fig. 1.

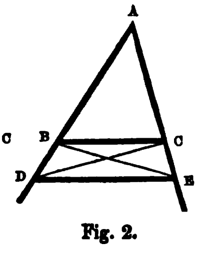


Fig. 2.

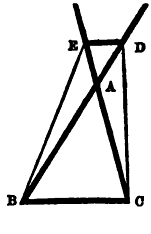


Fig. 3.

1), or these produced either way (Figs 2, 3), in D, E. Then they shall be cut proportionally; that is, BD shall be to DA as CE is to EA, or AD to DB as AE to EC.

Join BE, CD.

Because the triangles DEB, DCE are on the same base DE and between the same parallels DE, BC, they are equal (i. 37). And ADE is another triangle; and two equal magnitudes have each of them the same ratio to a third magnitude of the same kind (v. 7): therefore the triangle BDE is to the triangle ADE as the triangle CDE is to the triangle ADE. Now, since the triangles BDE, ADE have the same altitude, viz. the perpendicular drawn from E to AB, or AB produced, therefore BD (vi. 1) is to DA as the triangle BDE to the triangle ADE, and for like reason CE is to EA as the triangle CDE is to the triangle ADE; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore BD is to DA as CE to EA, and, invertendo (v. B), AD is to DB as AE to EC. Which was to be proved.

II. Let the two sides AB, AC of the triangle ABC (Fig. 1) or the sides produced either way (Figs. 2, 3) be cut proportionately in D, E; that is, let BD be to DA as CE is to EA, or invertendo, AD to DB as AE to EC. Then if these points D, E be joined, DE shall be parallel to BC.

Construct as in Part I. of the propn.

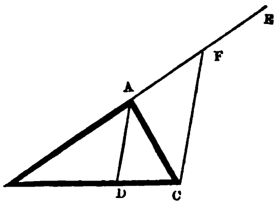
Then as before it may be shewn that BD is to DA as the triangle BDE to the triangle ADE, and CE to EA as the triangle CDE to the triangle ADE. But by hyps BD is to DA as CE to EA; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore the triangle BDE is to the triangle ADE as the triangle CDE to the triangle ADE. But two magnitudes which have each of them the same ratio to a third are equal (v. 9): therefore the triangle BDE is equal to the triangle CDE. Hence BDE, CDE are equal triangles on the same base DE and on the same side of it; therefore they are between the same parallels (i. 39), that is, DE is parallel to BC. Which was to be proved.

PROP. III. THEOR.

If an angle of a triangle be bisected by a straight line cutting the opposite side: then the two parts into which this side is divided shall have the same ratio that the two other sides of the triangle have to one another. And if a side of a triangle be divided into two parts, having the same ratio which the two other sides of the triangle have to one another: then the straight line drawn from the opposite angular point to the point of section shall bisect the angle.

I. Let ABC be a triangle, and let one of its angles BAC be bisected by AD, cutting the opposite side BC in D. Then shall BD be to DC as BA to AC.

Produce BA to E; and through C draw (i. 31) CF parallel to DA, cutting BE in F.



Because CA cuts the parallels AD, FC in A, C, the alternate angles DAC, ACF are equal (i. 29); and because BE cuts the parallels AD, FC in A, F, the exterior angle BAD is equal to the interior and opposite angle on the same side, AFC. But the angles BAD, CAD are equal by hyp^{s}; therefore likewise the angles ACF, AFC are equal, and therefore AC is equal (i. 6) to AF. Now since AD is drawn parallel to the side CF of the triangle BCF, cutting the two other sides BC, BF in D, A, BD is to DC as BA to AF (vi. 2); and it has been just shewn that AF is equal to AC: therefore BD is to DC as BA to AC. Which was to be proved.

II. Let ABC be a triangle; and let one of its sides BC be divided into two parts BD, DC in D, such that BD is to DC as BA to AC. Then if AD be joined, AD shall bisect the angle BAC.

Construct as in Part I. of the prop^{n}.

Then as before it may be shewn that the angles ACF, CAD, and the angles AFC, BAD are equal; and that BD is to DC as BA to AF. Now by hyp^{s} BD is to DC as BA to AC;

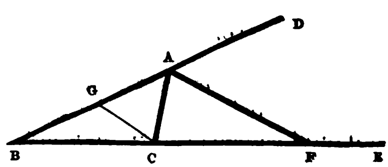
and ratios that are the same to the same ratio are the same to one another (v. 11): therefore BA is to AF as BA to AC. But two magnitudes to each of which a third has the same ratio are equal (v. 9); therefore AF is equal to AC, and therefore the angle ACF is equal (i. 5) to the angle AFC. And the angle ACF is equal to the angle CAD, and the angle AFC to the angle BAD: therefore likewise the angle CAD is equal to the angle BAD, that is, AD bisects the angle DAC. Which was to be proved.

PROP. A.

If two sides of a triangle be produced in the same direction, and one of the exterior angles thus formed be bisected by a straight line, cutting the opposite side produced: then the parts of this side produced intercepted between the bisecting line and the extremities of the side shall have the same ratio which the other two sides of the triangle have to one another. And if two sides of a triangle be produced in the same direction, and the part of one of them produced be cut in a point so that the parts of this side produced intercepted between the point of section and the extremities of the side have the same ratio to one another which the other two sides of the triangles have: then the straight line drawn from the opposite angular point to the point of section shall bisect the exterior angle at that point.

Let ABC be a triangle, and let two of its sides BA, BC be produced in the same direction to D, E; and :—

I. Let one of the exterior angles CAD be bisected by AF, cutting CE in F. Then BF shall be to FC as BA is to AC.



Through C draw (i. 31) CG parallel to FA, cutting AB in G.

Because CA cuts the parallels CG, FA in C, A, the alternate angles ACG,

CAF are equal (i. 29); and because BD cuts the parallels CG, FA in G, A the exterior angle FAD is equal to the interior and opposite angle on the same side, CGA. But the angles CAF, FAD are equal by hyp^s; therefore likewise the angles ACG, AGC are equal, and therefore AG is equal (i. 6) to AC. Now since AF is drawn parallel to the side GC of the triangle BGC, cutting the two other sides BG, BC produced in A, F, BF is to FC as BA to AG (vi. 2); and AG has been shewn to be equal to AC: therefore BF is to FC, as BA to AC. Which was to be proved.

II. Let the part of BC produced, CE, be cut in the point F, so that BF is to FC as BA to AC. Then if AF be joined, AF shall bisect the exterior angle CAD.

Construct as in Part I. of the prop^n.

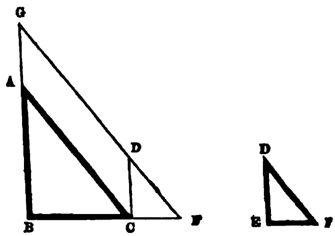
Then as before it may be shewn that the angles ACG, CAF and the angles FAD, CGA are equal; and that BF is to FC as BA to AG. Now by hyp^s BF is to FC as BA to AC; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore BA is to AG as BA to AC. But two magnitudes to each of which a third has the same ratio are equal (v. 9); therefore AG is equal to AC, and therefore the angle ACG (i. 5) to the angle AGC. And the angle ACG is equal to the angle CAF, and the angle AGC to the angle DAF; therefore likewise the angle CAF is equal to the angle DAF, that is, AF bisects the exterior angle CAD. Which was to be proved.

PROP. IV. THEOR.

If two triangles have the three angles of the one respectively equal to the three angles of the other: then they shall be similar, i. e. the sides also about each pair of the equal angles shall be proportional, those which are opposite to the equal angles being homologous in each proportion.

Let the two triangles ABC, DEF have the angles ABC, BCA respectively equal to the angles DEF, EFD, and consequently (i. 32. Cor. A) the third angle BAC to the third angle EDF. Then these triangles shall be similar; i. e. the sides also about each pair of equal angles shall be propor-

tional, those which are opposite to the equal angles being homologous in each proportion, viz. AB shall be to BC as DE is to EF, BC to CA as EF to FD, and BA to AC as ED to DF.



Apply the triangle DEF to the triangle ABC, so that the two sides BC, EF, opposite to a pair of equal angles BAC, EDF may be contiguous and in one straight line, and that neither pair of equal angles may be adjacent to one another both triangles falling on the same side of BCF. And let DCF be the position which it assumes, the point E coinciding with the point F. Since the angle DFC is equal to the angle ACB by hyp^s, to each of these equals add the angle ABC; then the angles DFC, ABC are equal (Ax. 2) to the angles ACB, ABC; but the two angles ACB, ABC of the triangle ABC are less (i. 17) than two right angles: therefore the angles ABC, DFC are less than two right angles. Hence the straight line BF, cutting the two straight lines BA, FD in B, F, makes the two interior angles on the same side of BF together less than two right angles: therefore by the axiom (Ax. 12), BA, FD being continually produced, shall at length meet in some point on the side of BF towards A, D. Let them be produced to meet in G.

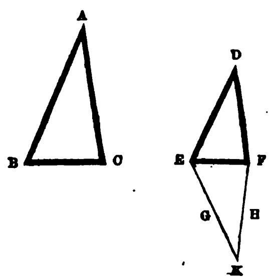
Now because BF, cutting BG, CD in B, C, makes the interior angle DCF equal to the interior and opposite angle on the same side, ABC; therefore GB is parallel (i. 28) to DC; and because BF cutting AC, GF in C, F makes the exterior angle ACB equal to the interior and opposite angle on the same side GFB, therefore AC is parallel to GF. Hence GACD is a parallelogram (Def. A); and the opposite sides of parallelograms are equal (i. 34): therefore GA is equal to DC, and GD to AC. Now since AC is drawn parallel to the side GF of the triangle GBF, cutting the two other sides GB, FB in A, C, therefore BA is to AG as BC to CF (vi. 2); and AG is equal to CD: therefore BA is

to CD as BC to CF. Hence, alternando (V. 16), AB is to BC as DC to CF, that is, AB is to BC as DE to EF. Again since CD is drawn parallel to the side BG of the triangle FBG, cutting the two other sides FB, FG in C, D, therefore BC is to CF as GD to DF; and GD is equal to AC: therefore BC is to CF as AC to DF. Hence, alternando, BC is to CA as CF to FD, that is, BC is to CA as EF to FD. Lastly, since AB is to BC as DE to EF, and BC is to CA as EF to FD; therefore, ex æquali (V. 22), BA is to AC as ED to DF. Hence the sides of the two triangles about each of the three pairs of equal angles have been shewn to be proportional, those opposite to equal angles being homologous. Which was to be proved.

PROP. V. THEOR.

If two triangles have their sides about each of the three pairs of angles proportional: then they shall be similar, i.e. the three angles also of the one shall be respectively equal to the three angles of the other, those being the equal angles to which the homologous sides in the proportions are opposite.

Let the two triangles ABC, DEF have their sides about each of the three pairs of angles proportional, viz. let AB be to BC as DE is to EF, BC to CA as EF to FD, and consequently, ex æquali (V. 22), BA to AC as ED to DF. Then these two triangles shall be similar, i.e. the angles also, opposite to the homologous sides in the proportions shall be equal, viz. ABC to DEF, BCA to EFD, and BAC to EDF.



At the points E, F in the straight line EF make (I. 23) the angle FEG equal to the angle ABC, and the angle EFH equal to the angle. Then adding equals to equals, the angles FEG, EFH are together equal (Ax. 2) to the angles ABC, ACB of the triangle ABC; and it may be shewn as in Prop. IV. that EG, FH, being

continually produced, shall at length meet in some point on the side of EF towards G, H. Let them be produced to meet in K.

The two triangles ABC, KEF have by const[n] the two angles ABC, ACB respectively equal to the two angles DEF, DFE, and consequently (i. 32. Cor. A) the third angle BAC equal to the third angle EKF; therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence AB is to BC as KE to EF, and BC to CA as EF to FK. But by hyp[s] AB is to BC as DE to EF; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore KE is to EF as DE to EF. And two magnitudes, which have each of them the same ratio to a third, are equal (v. 9): therefore KE is equal to DE. By like reasoning KF is equal to DF, and EF is common to the two triangles DEF, KEF; therefore these two triangles have the three sides DE, EF, FD respectively equal to the three sides KE, EF, FK. Therefore they are equal in every respect (i. 8); and hence the three angles DEF, EFD, FDE are equal to the three angles KEF, EFK, FKE. But these three are equal to the three angles ABC, BCA, CAB; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the three angles ABC, BCA, CAB are respectively equal to the three angles DEF, EFD, FDE. Which was to be proved.

PROP. VI. THEOR.

If two triangles have

(1) one angle of the one equal to one angle of the other;

(2) the sides about this pair of angles proportional:

then these two triangles shall be similar, i. e.

(1) the remaining angles also of the one shall be respectively equal to the remaining angles of the other, those being the equal angles to which the homologous sides in the proportion are opposite;

(2) the sides about each pair of these equal angles shall be proportional.

Let ABC, DEF be two triangles, which have

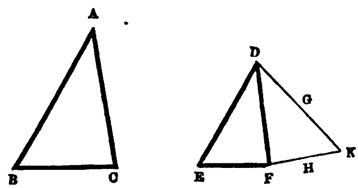
(1) the angle BAC of the one equal to the angle EDF of the other;

(2) the sides about the angles BAC, EDF proportional, viz. BA to AC as ED to DF.

Then these two triangles shall be similar, i. e.

(1) the remaining angles ABC, ACB of the one shall be respectively equal to the remaining angles DEF, DFE of the other, viz. ABC to DEF, to which the homologous sides AC, DF are opposite, and ACB to DFE, to which the homologous sides AB, ED are opposite;

(2) the sides about each pair of these equal angles shall be proportional, viz. AB shall be to BC as DE is to EF, and BC to CA as EF to FD.



At the points D, F in the straight line DF make (i. 23) the angle FDG equal to the angle BAC or EDF, and the angle DFH equal to the angle ACB. And as in the preceding propn, let DG, FH be produced to meet in K.

The two triangles ABC, DKF have by constn the two angles BAC, ACB respectively equal to the two angles KDF, DFK, and consequently (i. 32. Cor. A) the third angle ABC equal to the third angle DKF; therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence BA is to AC as KD to DF. But by hyps BA is to AC as ED to DF; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore ED is to DF as KD to DF. And two magnitudes which have each of them the same ratio to a third, are equal (v. 9): therefore ED is equal to DK. Also by constn the angle EDF is equal to the angle KDF,

and DF is common to the two triangles EDF, KDF; therefore these two triangles have the two sides ED, DF respectively equal to the two sides KD, DF, and the included angle EDF equal to the included angle KDF. Therefore they are equal in every respect (i. 4); and hence the angle DEF is equal to the angle DKF and the angle DFE to the angle DFK. But the angle DKF is equal to the angle ABC, and the angle DFK to the angle ACB; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angle ABC is equal to the angle DEF, and the angle ACB to the angle DFE. Lastly, since the triangles ABC, DEF have been shewn to be equiangular to one another, they are similar (vi. 4), and hence AB is to BC as DE to EF, and BC to CA as EF to FD. Which was to be proved.

PROP. VII. THEOR.

If two triangles have

(1) one angle of the one equal to one angle of the other;

(2) the sides about one of the other two pairs of angles proportionals;

and if each of the third angles be either less, or not less than a right angle, or if one of them be a right angle: then these two triangles shall be similar, i. e.

(1) the remaining angles also of the one shall be equal to the remaining angles of the other, those being the equal angles, the sides about which are proportional;

(2) the sides about the other two pairs of angles shall be proportional.

Let ABC, DEF be two triangles, which have

(1) the angle BAC of the one equal to the angle EDF of the other;

(2) the sides about one of the two other pairs of angles, ABC, DEF, proportional, viz. AB to BC as DE to EF;

and let each of the third angles ACB, DFE be either less,

or not less than a right angle, or one of them be a right angle. Then these two triangles shall be similar, i. e.

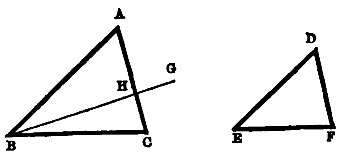
(1) the remaining angles ABC, ACB of the one shall also be respectively equal to the remaining angles DEF, EFD of the other;

(2) the sides about each pair of these equal angles shall be proportional, viz. AB shall be to BC as DE is to EF, and BC to CA as EF to FD.

There will be three cases according as each of the third angles is less than a right angle, or not less than a right angle, or one of them is a right angle.

I. Let each of the third angles, BCA, EFD be less than a right angle.

Then if the angles ABC, DEF be not equal; let them, if possible, be unequal, and let ABC be the one which is greater than the other DEF. At the point B in the straight line AB make (i. 23) the angle ABG equal to the angle DEF; and since BG must fall between BA and BC, let H be the point where it cuts AC.

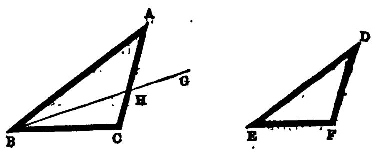


By constn the angle ABH is equal to the angle DEF, and by hyps the angle BAH to the angle EDF; hence the two triangles ABH, DEF have the two angles HBA, BAH respectively equal to the two angles FED, EDF, and consequently (i. 32. Cor. A) the third angle BAH equal to the third angle EDF: therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence AB is to BH as DE to EH. But by hyps AB is to BC as DE to EF; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore AB is to BH as AB to BC. And two magnitudes to each of which a third has the same ratio are equal (v. 9): therefore BH is equal to BC, and therefore the angle BHC (i. 5) to the angle BCH. Now the angle BCH is supposed less than a right angle; therefore also the angle BHC is less than a right angle. But be-

cause BH makes with AC on the same side of it the adjacent angles BHC, BHA, these two angles (I. 13) are equal to two right angles; and one of them BHC has been shewn to be less than a right angle: therefore the other BHA is greater than a right angle. And the angle DFE was proved to be equal to the angle BHA; therefore also the angle DFE is greater than a right angle. But by hyp[s] it is less than a right angle: which is impossible. Therefore the angles ABC, DEF are not unequal, that is, they are equal, and the angle BAC is equal to the angle EDF. Hence the two triangles ABC, DEF have the two angles CBA, BAC respectively equal to the two angles FED, EDF, and consequently the third angle ACB to the third angle DFE; therefore they are equiangular to one another.

II. Let each of the angles BCA, EFD be not less than a right angle.

If the angles ABC, DEF be not equal: let them, if possible, be unequal, and let ABC be the one which is greater than the other DEF.

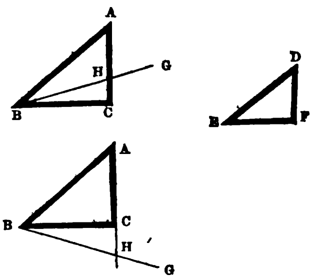


Construct as before.

Then as in Case I. it may be shewn that the angle BHC is equal to the angle BCH; but the angle BCH is supposed not less than a right angle: therefore the angle BHC is not less than a right angle. Hence the two angles BHC, BCH of the triangle BHC are together not less than two right angles: which is impossible (I. 17). And as before it may be shewn that the triangles ABC, DEF are equiangular to one another.

III. Let one of the angles BCA, EFD, viz. the angle BCA be a right angle.

If the angles ABC, DEF be not equal: let them, if possible, be unequal. At the point B in the straight line BA make the angle ABG equal to the angle DEF; and BG must cut either AC or AC produced. Let it cut AC or AC produced in H.



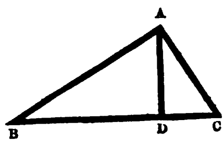
Then as in Case I. it may be shewn that the angle BHC is equal to the angle BCH; but the angle BCA is supposed a right angle, and accordingly BCH in the figure where AC is produced is one also (Def. 10): therefore the angle BHC is a right angle. Hence the two angles BHC, BCH of the triangle BHC are equal to two right angles: which is impossible (i. 17). And as in Case I. it may be shewn that the triangles ABC, DEF are equiangular to one another.

Therefore in all three cases, the two triangles are equiangular to one another; therefore they are similar (vi. 4), and hence the angles ABC, ACB are respectively equal to the angles DEF, DFE, AB is to BC as DE to EF, and BC to CA as EF to FD. Which was to be proved.

PROP. VIII. THEOR.

If a right-angled triangle be divided into two triangles by a perpendicular drawn from the right angle to the opposite side: then these two triangles shall be similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from A let AD be drawn perpendicular to the opposite side BC, dividing the triangle ABC into the two triangles ABD, ADC. Then these two triangles shall be similar to the triangle ABC, and to one another.



By hyp[s] each of the angles BDA, BAC is a right angle; and all right angles are equal (Ax. 11): therefore the angles BDA, BAC are equal. Also the angle at B is common to the two triangles ABD, CBA: hence these two triangles have the two angles

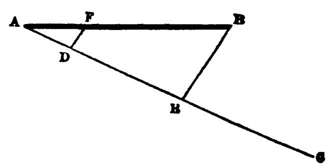
ADB, DBA respectively equal to the two angles CAB, ABC, and consequently (i. 32. Cor. A) the third angle BAD equal to the third angle BCA, that is, they are equiangular to one another. Therefore the triangle ABD is similar (vi. 4) to the triangle ABC. In like manner it may be shewn that the triangle ACD is equiangular, and therefore similar, to the triangle ABC. Lastly, since the three angles of each of the triangles ABD, ACB are respectively equal to the three angles of the triangle ABC; and things that are equal to the same things are equal to one another (Ax. 1): therefore the triangles ABD, ACB are equiangular to one another, and therefore similar. Which was to be proved.

COR.—The perpendicular drawn from the right angle of a right-angled triangle to the opposite side shall be a mean proportional between the two parts into which it divides this side; and each of the sides including the right angle shall be a mean proportional between the opposite side, and the part of it adjacent to that side.

For by the prop^n the triangles ADB, ADC are similar, and hence (vi. Def. 1) BD is to DA as DA to DC; that is, DA is a mean proportional (v. Def. 9. Obs. 2) between BD and DC. Also by the prop^n the triangles ABC, ABD are similar, and hence BC is to BA as BA to BD; that is, AB is a mean proportional between BC and BD, and in like manner AC is a mean proportional between BC and CD. Which was to be proved.

PROP. IX. PROB.

From a given straight line to cut off a part (i. e. measure or submultiple), which shall be contained in the whole line a given number of times.



Let AB be the given straight line. It is required to cut off from AB a part, i. e. measure or submultiple (v. Def. 1),

which shall be contained in the whole AB a given number of times.

Through A draw any straight line AC, making an angle BAC with AB; in AC take any point D; and from AC cut off (i. 3) AE equal to the same multiple of AD that AB is of the part to be cut off from it. Join BE; and through D draw (i. 31) DF parallel to EB, cutting AB in F. Then AF shall be the part required.

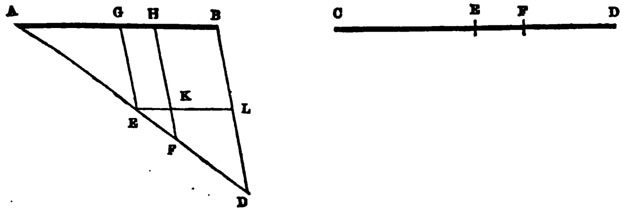
Since by constⁿ DF is drawn parallel to the side EB of the triangle ABE, cutting the two other sides AE, AB in D, F, therefore ED is to DA as BF to FA (vi. 2); and, componendo (v. 18), AE is to AD as AB to AF. Now, by constⁿ AE is a multiple of AD, viz. the same that AB is of the part to be cut off from it; and if four magnitudes be proportional, and the first be a multiple of the second, the third is the same multiple of the fourth (v. D): therefore AB is the same multiple of AF that AB is of the part required to be cut off from it. Hence from the given straight line AB the part required, AF, has been cut off. Which was to be done.

PROP. X. PROB.

To divide a given straight line proportionally to a given divided straight line; i. e. into parts that shall have two and two the same ratios to one another which the parts two and two of the divided line have.

Let AB be the given straight line to be divided, and CD the given divided straight line, E, F being the points of section. It is required to divide AB proportionally to CD.

Let CD be placed so as to contain any angle with AB,



and let AEFD be the position which it assumes, the point C coinciding with the point A. Join DB; and through E, F draw (i. 31) EG, FH parallel to DB and cutting AB in G, H. Then AB shall be divided in G, H, proportionally to CD.

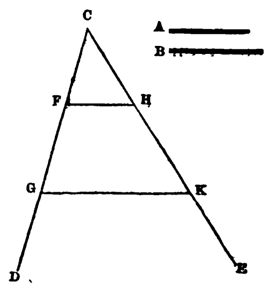
Through E draw EKL parallel to AB, cutting HF in K, and BD in L.

By const^n EH, BK are each of them parallelograms; and the opposite sides of parallelograms are equal (i. 34): therefore EK is equal to GH, and KL to HB. Now since GE is drawn parallel to the side HF of the triangle AHF, cutting the other two sides AH, AF in G, E, therefore AG is to GH as AE to EF (vi. 2). Again, since KF is drawn parallel to the side LD of the triangle ELD cutting the two other sides EL, ED in K, F, therefore EK is to KL as EF to ED; but EK is equal to GH, and KL to HB: therefore GH is to HB as EF to FD. Hence, since AG is to GH as CE to EF, and GH to HB as EF to FD, the given straight line AB has been divided in G, H proportionally to CD. Which was to be done.

PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let A, B be the two given straight lines. It is required to find a third proportional to A, B.



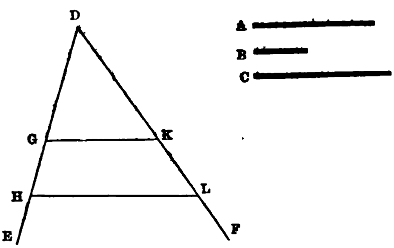
Take two straight lines CD, CE, including an angle DCE; from CD cut off (i. 3) CF equal to A; from FD cut off FG equal to B; and from CE cut off CH equal to B. Join FH, and through G draw (i. 31) GK parallel to FH, cutting CE in H. Then KH shall be a third proportional to A and B.

Since FH by const^n is drawn parallel to the side GK of the triangle CGK, cutting the two other sides CG, CK in F, H, therefore CF is to FG as CH to

HK (vi. 2); but by constⁿ CF is equal to A, FG to B, and CH to B: therefore A is to B as B to HK. Hence (v. Def. 9) to the two given straight lines A, B has been found a third proportional HK. Which was to be done.

PROP. XII. PROB.

To find a fourth proportional to three given straight lines.



Let A, B, C be the three given straight lines. It is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF including an angle EDF; from DE cut off (i. 3) DG equal to A, from GE, GH equal to B; and from DF, DK equal to C. Join GK; and through H draw (i. 31) HL parallel to GK, cutting DF in L. Then KL shall be a fourth proportional to A, B, C.

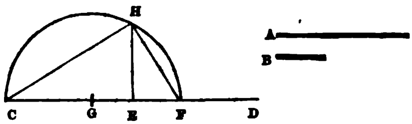
Since by constⁿ GK is drawn parallel to the side HL of the triangle DHL, cutting the two other sides DH, HL in G, K, therefore DG is to GH as DK to KL (vi. 2); but by constⁿ DG is equal to A, GH to B, DK to C: therefore A is to B as C to KL. Hence (v. Def. 6. Obs.) to the three given straight lines A, B, C has been found a fourth proportional KL. Which was to be done.

PROP. XIII. PROB.

To find a mean proportional between two given straight lines.

Let A, B be the two given straight lines. It is required to find a mean proportional between them.

Take any straight line CD; and from it cut off (i. 3) CE equal to A, and from ED, EF equal to B. Bisect (i. 10) CF in G, and with centre G and radius GC or CF describe a semicircle. From E draw EH at right angles to CF, cut-



ting the circumference in H. Then EH shall be a mean proportional between A and B.

Join HC, HF.

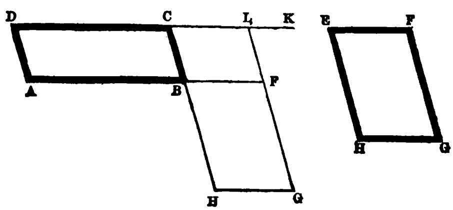
The angle CHF is an angle in a semicircle, and therefore (iii. 31) a right angle; and because in the right-angled triangle CHF, HE is drawn perpendicular from the right angle H to the opposite side CF, therefore HE is a mean proportional (vi. 8. Cor.) between CE, EF, the two parts of CF. But by const^n CE is equal to A, and EF to B: hence between the two given straight lines A, B has been found a mean proportional HE. Which was to be done.

PROP. XIV. THEOR.

If two parallelograms, which have one angle of the one equal to one angle of the other, be equal: then they shall have their sides about this pair of equal angles reciprocally proportional. And if two parallelograms, which have one angle of the one equal to one angle of the other, have the sides about this pair of equal angles reciprocally proportional: then they shall be equal.

Let ABCD, EFGH be two parallelograms, having the angle ABC equal to the angle FEH; and:—

I. Let ABCD be equal to EFGH. Then they shall have their sides about the pair of equal angles ABC, FEH reciprocally proportional; that is, AB shall be to EF as EH is to BC.



Apply EFGH to ABCD, so that the sides AB, EF may be contiguous and in one straight line ABF, and the sides CB, EH may fall on opposite sides of ABF; and let BFGH be the position which it assumes, the point E coinciding with the point B. Now, since by hyp[s] the angle ABC is equal to the angle FBH, to each of these equals add the angle CBF: then the angles ABC, CBF are equal (Ax. 2) to the angles HBF, CBF. But because CB makes with AF on the same side of it the adjacent angles ABC, CBF, these two angles are equal (i. 13) to two right angles; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the angles HBF, FBC are equal to two right angles. That is, at the point B in the straight line BF, the two straight lines CB, HB make with BF, on opposite sides of it, the adjacent angles CBF, HBF equal to two right angles: therefore CB is in the same straight line (i. 14) with BH. Also produce DC to K, and GF to cut CK in L.

By const[n] CBFL is a parallelogram, and the two parallelograms DB, BG are equal by hyp[s]; and two equal magnitudes have each of them the same ratio to a third of the same kind (v. 7): therefore DB is to CF as BG is to CF. But because the parallelograms DB, CF have the same altitude, viz. the perpendicular drawn from C to AF, or AF produced, therefore DB is to CF as AB to BF (vi. 1). In like manner BG is to CF as HB to BC; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore AB is to BF as HB to BC, that is, AB is to EF as EH to BC. Which was to be proved.

II. Let ABCD, EFGH have their sides about the pair of equal angles ABC, FEH reciprocally proportional; that is, let AB be to EF as EH is to BC. Then they shall be equal.

Construct as in Part I. of the prop[n].

Then, as before, it may be shewn that DB is to CF as AB to BF, and BG to CF as HB to BC. But by hyp[s] AB is to BF as HB to BC; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore DB is to CF as BG is to CF. And two magnitudes which have each of them the same ratio to a third are

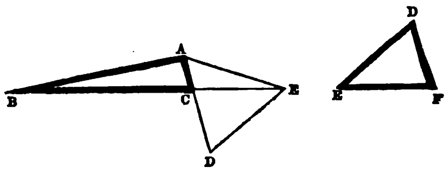
equal (v. 9): therefore DB is equal to BG, that is, the parallelogram ABCD is equal to the parallelogram EFGH. Which was to be proved.

PROP. XV. THEOR.

If two triangles, which have one angle of the one equal to one angle of the other, be equal: then they shall have their sides about this pair of equal angles reciprocally proportional. And if two triangles, which have one angle of the one equal to one angle of the other, have the sides about this pair of equal angles reciprocally proportional: then they shall be equal.

Let ABC, DEF be two triangles, having the angle ACB equal to the angle DFE; and:—

I. Let ABC be equal to DEF. Then they shall have their sides about the pair of equal angles ACD, DFE reciprocally proportional; that is, BC shall be to EF as FD is to CA.



Apply DEF to ABC, so that the sides BC, FE may be contiguous and in one straight line BCE, and the sides AC, FD may fall on opposite sides of BCE; and let DCE be the position which it assumes, the point F coinciding with the point C. Then it may be shewn exactly as in the preceding propⁿ, that AC, CD are in one straight line. Join AE.

By hypˢ the triangles ACB, DCE are equal; and two equal magnitudes have each of them the same ratio to a third of the same kind (v. 7): therefore ABC is to the triangle ACE as DCE is to ACE. But because the triangles ABC, ACE have the same altitude, viz. the perpendicular drawn from A on BE, or BE produced, therefore ABC is to ACE as BC to CE (vi. 1). In like manner DCE is to ACE

as DC to CA; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore BC is to CE as DC to CA, that is, BC is to EF as FD to CA. Which was to be proved.

II. Let ABC, DEF have their sides about the pair of equal angles ACB, DFE reciprocally proportional; that is, let BC be to EF as FD is to CA. Then they shall be equal.

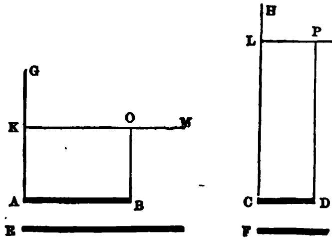
Construct as in Part I. of the prop^n.

Then, as before, it may be shewn that ABC is to ACE as BC to CE, and DCE to ACE as DC to CA. But by hyp^s BC is to CE as DC to CA; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore ABC is to ACE as DCE to ACE. And two magnitudes which have each of them the same ratio to a third are equal (V. 9): therefore ABC is equal to CDE, that is, the triangle ABC is to the triangle DEF. Which was to be proved.

PROP. XVI. THEOR.

If four straight lines be proportional: then the rectangle contained by the two extremes shall be equal to that contained by the two means. And if there be four straight lines, the rectangle contained by the two extremes of which is equal to that contained by the two means: then they shall be proportional.

I. Let the four straight lines AB, CD, E, F be proportional; AB, F being the two extremes, and CD, E the two means (V. Def. 6. Obs.). Then the rectangle AB, F shall be equal to the rectangle CD, E.



From A, C draw (I. 11) AG, CH at right angles to AB, CD; from AG cut off (I. 3) AK equal to F, and from CH cut off CL equal to E; through K, L draw (I. 31) KM, LN parallel to AB, CD, and through B, D draw BO, DP parallel to AG, CH, cutting KM, LN in O, P.

The figures KB, LD are parallelograms by const^{n}, and since they contain the right angles KAB, LCD, therefore they are each of them rectangles (i. 46. Cor.). Also KB is the rectangle AB, F, for it is contained by AB, AK, and AK is equal to F by const^{n}; in like manner LD is the rectangle CD, E. Now, since by hyp^{s} AB is to CD as E to F; and E is equal to CL, and F to AK: therefore AB is to CD as CL to AK. That is, the two parallelograms KB, LD, having the equal (Ax. 11) angles KAB, LCD, have the sides about these angles reciprocally proportional: therefore KB is equal to LD (vi. 14), that is, the rectangle AB, F is equal to the rectangle CD, E. Which was to be proved.

II. Let AB, CD, E, F be four straight lines; AB, F the two extremes; CD, E the two means; and let the rectangle AB, F be equal to the rectangle CD, E. Then AB shall be to CD as E is to F.

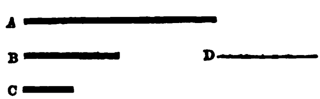
Construct as in Part I.

Then, as before, it may be shewn that KB is the rectangle AB, F, and LD is the rectangle CD, E: hence by hyp^{s} KB is equal to LD. That is, the two parallelograms KB, LD, having the angle KAB equal to the angle LCD, are equal; therefore they have their sides about these angles reciprocally proportional (vi. 14), that is, AB is to CD as CL to AK. But by const^{n} CL is equal to E, and AK to F; therefore AB is to CD as E to F. Which was to be proved.

PROP. XVII. THEOR.

If three straight lines be proportional: then the rectangle contained by the two extremes shall be equal to the square of the mean. And if there be three straight lines, the rectangle contained by the two extremes of which is equal to the square of the mean: then they shall be proportional.

I. Let A, B, C be three straight lines, and let A be to B as B is to C. Then the rectangle contained by A, C shall be equal to the square of B.



Take a straight line D equal to B.

By hyp[s] A is to B as B is to C, and D is equal to B; therefore A is to B as D is to C. But if four straight lines be proportional, the rectangle contained by the extremes is equal to that contained by the means (vi. 16); therefore the rectangle A, C is equal to the rectangle B, D; and the rectangle B, D is the square of B, for B is equal to D: therefore the rectangle A, C is equal to the square of B. Which was to be proved.

II. Let A, B, C be three straight lines, and let the rectangle contained by A, C be equal to the square of B. Then A shall be to B as B is to C.

Construct as in Part I.

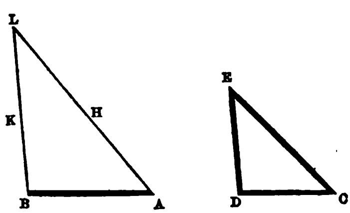
Then, as before, the square of B is equal to the rectangle B, D; and therefore the rectangle A, C is equal to the rectangle B, D. But four straight lines, the rectangle contained by the extremes of which is equal to that contained by the means, are proportional; therefore A is to B as D to C, that is, since D is equal to B, A is to B as B to C. Which was to be proved.

PROP. XVIII. PROB.

On a given straight line to describe a polygon, which shall be similar to a given polygon, and similarly situated with it; i. e. having its side which coincides with the given straight line homologous to a given side in the given polygon.

I. Let the polygon be a triangle.

Let AB be the given straight line, and CDE the given triangle. It is required to describe on AB a triangle similar to the triangle CDE, and similarly situated with it, viz. having its side AB homologous to a given side CD of the triangle CDE.

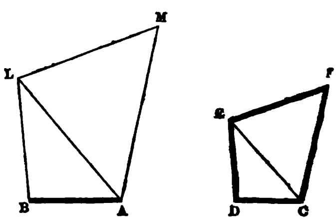


At the points A, B make (i. 23) the angles BAH, ABK respectively equal to the angles DCE, CDE; and, as in the const^n of Prop. V., AH, BK, being continually produced will meet: let them be produced to meet in L. Then ABL shall be the triangle required.

The two angles BAL, ABL of the triangle ABL are respectively equal to the two angles DCE, CDE of the triangle CDE, and consequently (i. 32. Cor. A) the third angle ALB is equal to the third angle CED; therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence BA is to AL as DC to CE, and BA, DC are homologous sides. Hence on the given straight line AB has been described a triangle ABL, similar to the given triangle CDE, and similarly situated with it, viz. having its side AB homologous to the given side CD of the triangle CDE. Which was to be done.

II. Let the polygon have four sides.

Let AB be the given straight line, and CDEF the given polygon of four sides. It is required to describe on AB a polygon similar to the polygon CDEF, and similarly situated with it, viz. having its side AB homologous to a given side CD of CDEF.



Join C with the opposite angular point E of CDEF, so as to divide it into the two triangles CDE, CEF. By the first case on AB describe the triangle ABL similar to the triangle CDE, and similarly situated with it, viz. having its side AB homologous to the side CD; and on AL describe the triangle ALM similar to the triangle CEF, and similarly situated with it, viz. having its side AL homologous to the side CE. Then the polygon ABLM shall be the polygon required.

Because the triangles ABL, CDE are similar by const^n, therefore (vi. Def. 1) the angle ABL is equal to the angle CDE, the angle BLA to the angle DEC, and the angle LAB

to the angle ECD; and because the triangles ALM, CEF are similar by constⁿ, therefore the angle ALM is equal to the angle CEF, the angle LMA to the angle EFC, and the angle MAL to the angle FCE. Now since the angle BLA is equal to the angle DEC, and the angle ALM to the angle CEF: therefore, adding equals to equals, the whole angle BLM is equal (Ax. 2) to the whole angle DEF: in like manner, the angle BAM is equal to the angle DCF. Hence the angles ABL, BLM, LMA, MAB of the polygon ABLM are respectively equal to the angles CDE, DEF, EFC, FCD, of the polygon CDEF. Again from the pair of similar triangles ABL, CDE, AB is to BL as CD to DE, BL to LA as DE to EC, and LA to AB as EC to CD; and from the pair of similar triangles ALM, CEF, AL is to LM as CE to EF, LM is to MA as EF to FC, and MA to AL as FC to CD. Now since BL is to LA as DE to EC, and LA is to AM as EC to CF, therefore, ex æquali (V. 22), BL is to LM as DE to EF; in like manner MA is to AB as FC to CD. Hence the sides about each pair of equal angles in the two polygons ABLM, CDEF are proportional, viz. AB to BL as CD to DE, BL to LM as DE to CF, LM to MA as EF to FE, and MA to AB as FC to CD; and BA, DC are homologous sides. Hence on the given straight line AB has been described a polygon ABLM similar to the given polygon CDEF of four sides, and similarly situated with it, viz. having its side AB homologous to the given side CD of CDEF.

III. Let the polygon have five or more sides.

The polygon must be divided into triangles by joining one of the extremities of the given side in it with each of its other angular points; and successive triangles be described on the given straight line and on the sides of one another, similar to the different triangles of the polygon, and similarly situated with them, just as ABL was on AB, and ALM on AL in Case II. And it can be shewn by a like proof that the resulting figure is a polygon fulfilling the requisites of the problem.

PROP. XIX. THEOR.

Similar triangles shall be to one another in the duplicate ratio of the ratio of either pair of their homologous sides.

Let ABC, DEF be two similar triangles, having the angles A, B, C respectively equal to the angles D, E, F, and AB to BC as DE to EF, BC to CA as EF to FD so that BC and EF, CA and FD, AB and DE are the three pairs of homologous sides. Then the triangle ABC shall have to the triangle DEF the duplicate ratio of the ratio which BC has to EF, or of that which CA has to FD, or of that which AB has to DE.

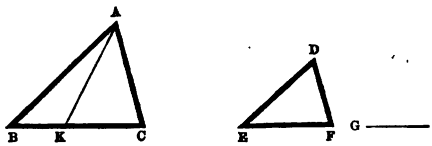


Fig. 1.

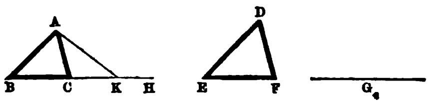


Fig. 2.

To BC, EF find (vi. 11) a third proportional G, so that BC is to EF as EF to G; from BC (Fig. 1), or BC produced if necessary to H (Fig. 2), cut off BK equal to G; and join AK.

By hyp[s] AB is to BC as DE to EF; and therefore, alternando (v. 16), AB is to DE as BC to EF. But by const[n] BC is to EF as EF to G, that is, as EF to BK, since BK is equal to G; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore AB is to DE as EF to BK. Hence the triangles ABK, DEF, which have by hyp[s] the angle ABK equal to the angle DEF, have their sides about these angles reciprocally proportional; therefore the triangle ABK is equal (vi. 15) to the triangle DEF. Again since the triangles ABC, ABK have

the same altitude, viz. the perpendicular drawn from A on BC, or BC produced, therefore the triangle ABC is to the triangle ABK as BC is to BK (VI. 1); but the triangle ABK has been shewn to be equal to the triangle DEF, and BK is equal to G: therefore the triangle ABC is to the triangle DEF as BC to G. Now since G has been taken a third proportional to BC and EF, the ratio of BC to G by the def[n] of duplicate ratio (V. Def. 10) is the duplicate ratio of the ratio which BC has to EF: therefore the triangle ABC has to the triangle DEF the duplicate ratio of the ratio which BC has to EF. And in like manner it may be shewn that the triangle ABC has to the triangle DEF the duplicate ratio of the ratio which CA has to FD, or of the ratio which AB has to DE. Which was to be proved.

COR.—If three straight lines be proportional: then the first shall be to the third as any triangle on the first is to the similar and similarly described triangle on the second.

For let the three straight lines BC, EF, G be proportional; and on BC, EF let ABC, DEF be two triangles similar and similarly described, viz. having BC, EF homologous sides. Then it was shewn in the prop[n] that BC is to G as the triangle ABC to the triangle DEF. Which was to be proved.

PROP. XX. THEOR.

If there be two similar polygons: then

(1) they may be divided into the same number of triangles, each pair of which shall be similar to one another;

(2) each pair shall have to one another the same ratio that the polygons have;

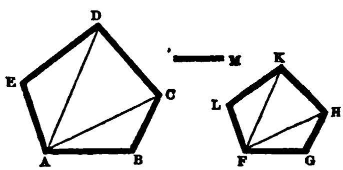
(3) the polygons shall have to one another the duplicate ratio of the ratio which either pair of their homologous sides have.

Let ABCDE, FGHKL be two similar polygons; having the angles A, B, C, D, E respectively equal to the angles F, G,

H, K, L and the sides AB, BC, CD, DE, EA respectively homologous to FG, GH, HK, KL, LF. Then:—

I. ABCDE, FGHKL may be divided into the same number of triangles, each pair of which shall be similar to one another.

Join A with each of the other angular points of ABCDE, and F with each of the other angular points of FGHKL. The polygons are thus divided into the same number of triangles.



Since by hyp[s] the polygons are similar, the angle ABC is equal to the angle FGH, and AB is to BC as FG to GH (vi. Def. 1); therefore the two triangles ABC, FGH have the angle ABC equal to the angle FGH, and the sides about this pair of angles proportional. Therefore these two triangles are similar (vi. 6); and hence the angle BCA is equal to the angle GHF, and AC is to CB as FH to HG:

Again, since the polygons are similar, the angle BCD is equal to the angle GHK, and BC is to CD as GH to HK. Now since the angle BCA is equal to the angle GHF, and the angle BCD to the angle GHK, therefore taking away equals from equals, the angle ACD is equal (Ax. 3) to the angle FHK; and since AC is to CB as FH to HG and BC is to CD as GH to HK, therefore, ex æquali (v. 22), AC is to CD as FH to HK: therefore the two triangles ACD, FHK have the angle ACD equal to the angle FHK, and their sides about this pair of angles proportional. Therefore these two triangles are similar; and hence the angle CDA is equal to the angle HKF, and CD is to DA as HK to KF:

In like manner it may be shewn that the two triangles DEA, KLF are similar.

Hence the two polygons ABCDE, FGHKL are divided each into the same number of triangles, viz. ABC, ACD, ADE and FGH, FHL, FKL, each pair of which are similar to one another, viz. ABC to FGH, ACD to FHK, and ADE to FKL. Which was to be proved.

II. Each pair of these similar triangles, viz. ABC and FGH, ACD and FHK, and ADE and FKL shall have to one another the same ratio that the polygons ABCDE, FGHKL have.

Because similar triangles have to one ratio the duplicate ratio of the ratio which either pair of their homologous sides have (VI. 19), the triangle ABC has to the triangle FGH the duplicate ratio of the ratio which AC has to FH, and the triangle ACD has to the triangle FHK the duplicate ratio of the ratio which AC has to FH; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore the triangle ABC is to the triangle FGH as the triangle ACD to the triangle FHK. In like manner it may be shewn that the triangle ACD is to the triangle FHK as the triangle ADE to the triangle FKL; and ratios that are the same to the same ratio are the same to one another: therefore the ratio of the triangle ABC to the triangle FGH, the ratio of the triangle ACD to the triangle FHK, and the ratio of the triangle ADE to the triangle FKL are the same to one another. But if any number of ratios be the same to one another any one of the antecedents is to its consequent as all the antecedents together are to all the consequents together (V. 12): therefore the triangle ABC is to the triangle FGH as the polygon ABCDE is to the polygon FGHKL, the triangle ACD is to the triangle FHK as the polygon ABCDE to the polygon FGHKL, and the triangle ADE is to the triangle FKL as the polygon ABCDE to the polygon FGHKL. Which was to be proved.

III. The polygon ABCDE shall have to the polygon FGHKL the duplicate ratio of the ratio which either of the pairs of homologous sides have to one another.

Each pair of homologous sides in the two polygons, as AB, FG is also a pair of homologous sides in one of the pairs of similar triangles ABC, FGH, into which in Part I. the polygons are divided. Now by Part II. the polygon ABCDE is to the polygon FGHKL as the triangle ABC to the triangle FGH. But since similar triangles have to one another the duplicate ratio of the ratio which either pair of their homologous sides have (VI. 19), the triangle

ABC has to the triangle FGH the duplicate ratio of the ratio which AB has to FG; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore the polygon ABCDE has to the polygon FGHKL the duplicate ratio of the ratio which AB has to FG. In like manner it may be shewn that the polygon ABCDE has to the polygon FGHKL the duplicate ratio of the ratio which BC has to GH, or of that which CD has to HK, or of that which DE has to KL, or of that which EA has to LF. Which was to be proved.

COR.—If three straight lines be proportional: then the first shall be to the third as any polygon on the first is to the similar and similarly described polygon on the second.

For let the three straight lines AB, FG, M, be proportional; and on AB, FG let ABCDE, FGHKL be two polygons similar and similarly described, viz. having AB, FG homologous sides. Then since M is a third proportional to AB and FG, AB has to M the duplicate ratio of the ratio which AB has to FG by the def^n of duplicate ratio (v. Def. 11). And by the prop^n, the polygon ABCDE has to the polygon FGHKL the duplicate ratio of the ratio which AB has to FG: hence, since ratios that are the same to the same ratio are the same to one another, AB is to M as the polygon ABCDE to the polygon FGHKL. Which was to be proved.

OBS. The third part of the prop^n and the above corollary prove that to be generally true for any polygon, which was shewn to be true in the particular case of triangles in Prop. 19 and its corollary.

COR. A.—If there be two similar polygons, the first shall be greater than, equal to, or less than the second, according as any side of the first is greater than, equal to, or less than the homologous side of the second; and conversely.

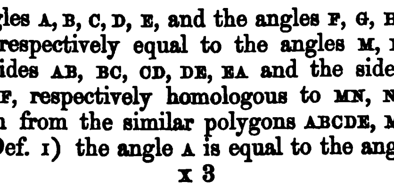
For if ABCDE, FGHKL be two similar polygons. the polygon ABCDE is to the polygon FGHKL as AB

to M, and AB is to FG as FG to M; AB, FG being a pair of homologous sides, and M being taken a third proportional (VI. 11) to AB and FG. Now if AB is equal to FG, then FG is equal (V. A) to M, and therefore, since things that are equal to the same thing are equal to one another, AB is equal to M; if AB is greater than FG, then FG is greater than M, and therefore by much more AB is greater than M; and if less, less. Hence AB is greater than, equal to, or less than M, according as AB is greater than, equal to, or less than FG; and ABCDE is greater than, equal to, or less than FGHKL, according as AB is greater than, equal to, or less than M: therefore ABCDE is greater than, equal to, or less than FGHKL, according as AB is greater than, equal to, or less than the homologous side FG. And by a method similar to that employed in Bk. I. Prop. XIX. or XXV. the converse may be shewn. Which was to be proved.

PROP. XXI. THEOR.

Two polygons which are each of them similar to a third polygon shall be similar to one another.

Let the two polygons ABCDE, FGHKL be each of them similar to a third polygon MNOPQ. Then the polygons ABCDE, FGHKL shall be similar to one another.



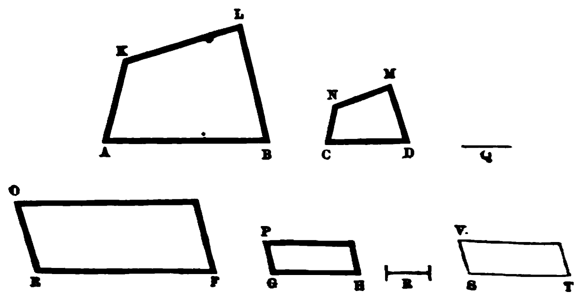
The angles A, B, C, D, E, and the angles F, G, H, K, L are supposed respectively equal to the angles M, N, O, P, Q; and the sides AB, BC, CD, DE, EA and the sides FG, GH, HK, KL, LF, respectively homologous to MN, NO, OP, PQ, QM. Then from the similar polygons ABCDE, MNOPQ, by def^n (VI. Def. 1) the angle A is equal to the angle M, and

EA is to AB as QM to MN; and from the similar polygons FGHKL, MNOPQ the angle F is equal to the angle M, and LF is to FG as QM to MN. Therefore since things that are equal to the same thing are equal to one another (Ax. I), the angle A is equal to the angle F; and since ratios that are the same to the same ratio are the same to one another (V. II), EA is to AB as LF to FG. In like manner it may be shewn that the angles B, C, D, E are respectively equal to the angles G, H, K, L; and that the sides about each pair of equal angles are proportional: hence by the def^n of similar polygons the polygons ABCDE, FGHKL are similar. Which was to be proved.

PROP. XXII. THEOR.

If four straight lines be proportional, and there be similarly described on the first and second any two similar polygons, and on the third and fourth any two similar polygons: then the polygon on the first shall be to that on the second as the polygon on the third is to that on the fourth. And if there be similarly described on the first and second of four straight lines two similar polygons, and on the third and fourth two similar polygons, and if the polygon on the first be to that on the second as the polygon on the third is to that on the fourth: then the four straight lines shall be proportional.

Let AB, CD, EF, GH be four straight lines; and on AB, CD, let ABLK, CDMN be two polygons similar and similarly



described, viz. having their sides AB, CD homologous; and on EF, GH let OF, PH be two polygons similar and similarly described, viz. having their sides EF, GH homologous. And :—

I. Let AB be to CD as EF is to GH. Then AKLB shall be to CNMD as OF is to PH.

To AB, CD find (vi. 11) a third proportional Q, and to EF, GH a third proportional R.

By constⁿ AB is to CD as CD to Q, and EF to GH as GH to R. But by hyp^s AB is to CD as EF to GH; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore CD is to Q as GH to R. Again, since AB is to CD as EF to GH; and CD is to Q as GH to R: therefore, ex æquali (v. 22), AB is to Q as EF to R. Now, if three straight lines are proportional, the first is to the third as any polygon on the first to the similar and similarly described polygon on the second (vi. 20. Cor.): therefore AB is to Q as AKLB to CNMD, and EF is to R as OF to PH. But AB has been shewn to be to Q as EF is to R; and ratios that are the same to the same ratio are the same to one another: therefore AKLB is to CNMD as OF to PH. Which was to be proved.

II. Let AKLB be to CNMD as OF is to PH. Then AB shall be to CD as EF is to GH.

To AB, CD, EF find (vi. 12) a fourth proportional ST; and on ST describe (vi. 18) the polygon VT similar either to OF or PH, and similarly situated with it, viz. having the side ST homologous either to the side EF or GH.

Since by $const^n$ AB is to CD as EF to ST, and there are similarly described on AB, CD the similar polygons AKLB, CMND, and on EF, ST the similar polygons OF, VT; therefore by Part I. of the $prop^n$ AKLB is to CNMD as OF to VT. But by hyp^s AKLB is to CNMD as OF to PH; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore OF is to VT as OF to PH; and two magnitudes to each of which the same magnitude has the same ratio are equal (v. 9): therefore PH is equal to VT. Now, PH is similar to VT, and similarly situated with it, viz. having the side GH homologous to ST; therefore GH is equal (vi. 20. Cor. A) to ST. But AB is to CD

as EF to ST, and ST is equal to GH; therefore AB is to CD as EF to GH. Which was to be proved.

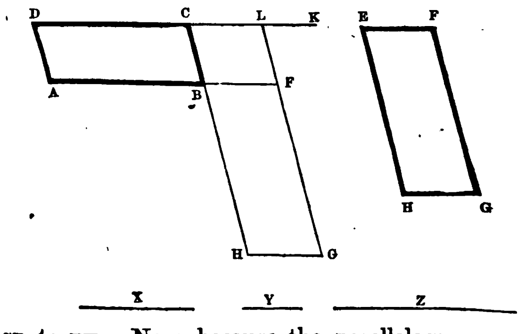
PROP. XXIII. THEOR.

If two parallelograms have one angle of the one equal to one angle of the other: then they shall have to one another the ratio which is compounded of the ratios of each pair of sides about the equal angles.

Let ABCD, EFGH be two parallelograms, having the angle ABC equal to the angle FEH. Then ABCD shall have to EFGH the ratio which is compounded of the two ratios of AB to EF, and of CB to EH.

Construct the figure exactly as in Prop. XIV.; and as in that prop^n, it may be shewn that CB, BH are in one straight line. Also take any straight line X; to AB, EF, X find (vi. 12) a fourth proportional Y; and to CB, EH, Y find a fourth proportional Z.

Then by const^n AB is to EF as X to Y, and CB is to EH as Y to Z; and therefore the ratio compounded of the two ratios of AB to EF, and of CB to EH, is the same as the ratio compounded of the two ratios of X to Y, and of Y to Z. But by the def^n of compound ratio (v. Def. 12), the ratio of X to Z is the ratio compounded of the two ratios of X to Y and of Y to Z; therefore the ratio of X to Z is the ratio compounded of the two ratios of AB to EB and

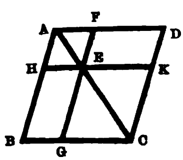


of CB to EH. Now, because the parallelograms DB, CF have the same altitude, viz. the perpendicular drawn from

C on AF, produced if necessary, DB is to CF as AB to BF (vi. 1); and AB is to BF as X to Y: therefore, since ratios that are the same to the same ratio are the same to one another (v. 11), DB is to CF as X to Y. In like manner it may be shewn that CF is to BG as Y to Z; therefore, ex æquali (vi. 22), DB is to BG as X to Z. But the ratio of X to Z was shewn to be the ratio compounded of the two ratios of AB to EB and of CB to EH: therefore the parallelogram ABCD has to the parallelogram EFGH the ratio which is compounded of the two ratios of AB to EF, and of CB to EC. Which was to be proved.

PROP. XXIV. THEOR.

If through any point in one of the diagonals of a parallelogram two straight lines be drawn parallel to its sides, so as to divide it into four parallelograms, two of which are about the diagonal of the parallelogram: then the two parallelograms about the diagonal shall be similar to the whole parallelogram and to one another.



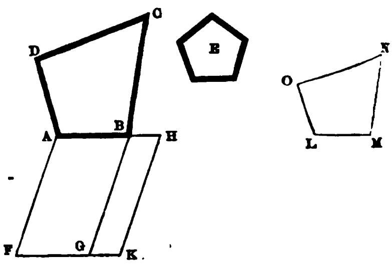
Let AC be one of the diagonals of the parallelogram ABCD, and E any point in AC; through E let FEG be drawn parallel to AB or CD, cutting AD in F and BC in G, and HEK parallel to AD or BC, cutting AB in H and CD in K, so as to divide ABCD into four parallelograms, of which the two HF, GK are about the diagonal AC. Then the parallelograms HF, GK shall be each of them similar to the parallelogram ABCD, and to one another.

Because AB cuts the parallels HE, BC in H, B, the exterior angle AHE is equal (i. 29) to the interior and opposite angle on the same side, ABC; for like reason the angle AFE is equal to the angle ADC. Also the angle DAB is common to the two parallelograms HF, BD; and since the opposite angles of parallelograms are equal (i. 34), each of the angles HEF, BCD is equal to it, and therefore (Ax. 1) to one another: therefore HF, BD are equiangular

to one another. Again, since the angle AHE is equal to the angle ABC, and the angle BAC common to the two triangles HAE, BAC; therefore these two triangles have the two angles AHE, HAE respectively equal to the two angles ABC, BAC, and consequently (i. 32. Cor. A) the third angle AEH equal to the third angle ACB: therefore they are equiangular to one another. Therefore they are similar (vi. 4); and hence AH is to HE as AB to BC. And since the opposite sides of parallelograms are equal, AH is equal to FE, HE to AF; AB to CD, and BC to AD. Hence HA is to AF as BA to AD, AF to FE as AD to DC, FE to EH as DC to CB: therefore the sides about each pair of equal angles of HF, BD are proportional. Therefore by the defⁿ of similar polygons (vi. Def. 1), HF is similar to BD. In like manner GK is similar to BD; and polygons, that are each of them similar to the same polygon, are similar to one another (vi. 21): therefore HF is similar to GK. Which was to be proved.

PROP. XXV. THEOR.

To describe a polygon, which shall be similar to one given polygon and equal to another given polygon.



Let ABCD be the given polygon to which the one to be described is required to be similar; and E that to which it is required to be equal. It is required to describe a polygon similar to ABCD and equal to E.

To one of the sides AB of ABCD apply (i. 45) the parallelogram AFGB, equal to the polygon ABCD, and having its angle AFG equal to any angle; and to the side BG of AG apply the parallelogram BHKG equal to the polygon E, and having the angle BGK equal to the angle AFG. Between AB and BH find (vi. 13) a mean proportional LM; and on

LM describe (vi. 18) the polygon LMNO, similar to ABCD and similarly situated with it, viz. having the side LM homologous to the side AB. Then LMNO shall be the polygon required.

It may be shewn exactly as in the proof of Bk. I. Prop. XLV. that FG, GK are in the same straight line, and that AB, BH are also. Hence the parallelograms AG, BK have the same altitude, viz. the perpendicular drawn from G on AH, or AH produced; therefore (vi. 1) AB is to BH as AG to BK. Now, by const^{n} AB is to LM as LM to BF; and if three straight lines be proportional the first is to the third as any polygon on the first to the similar and similarly described polygon on the second (vi. 20. Cor.): therefore AB is to BH as ABCD is to LMNO. But it was shewn that AB is to BH as AG to BK; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore ABCD is to LMNO as AG is to BK. Now, by const^{n} AG is equal to ABCD; therefore LMNO is equal (v. 14) to BK. But BK by const^{n} is equal to E; and things that are equal to the same thing are equal to one another (Ax. 1): therefore LMNO is equal to E, and it is similar to ABCD. Hence the polygon LMNO has been described similar to the given polygon ABCD and equal to the given polygon E. Which was to be done.

PROP. XXVI. THEOR.

If two similar parallelograms have a common angle, and the sides about this angle of the one fall on the homologous sides of the other: then their diagonals through this common angular point shall be in the same straight line.

Let the two similar parallelograms ABCD, AEFG have a common angle at A, and let the sides AE, AG about the angle EAG of AEFG fall on the homologous sides AB, AD of ABCD. Then the

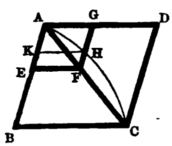


Fig. 1.

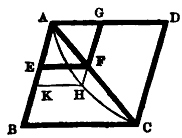


Fig. 2.

diagonals AF, AC through A shall be in the same straight line.

For if not: let, if possible, the diagonal of BD through A take some other direction than AF; and let GF (Fig. 1) or GF produced (Fig. 2) cut it in H. Through H draw (i. 31) HK parallel to AD or BC, cutting AB in K.

By constⁿ AKHG is a parallelogram about the diagonal AHC of the parallelogram ABCD, formed by drawing HK, HG parallel to its sides through a point H in the diagonal. Therefore AKHG is similar (vi. 24) to ABCD; and hence DA is to AB as GA to AK. But since GE, DB are similar by hyp', DA is to AB and GA to AE (vi. Def. 1); and ratios that are the same to the same ratio are the same to one another (v. 11): therefore GA is to AK as GA to AE. But two magnitudes to each of which a third has the same ratio are equal (v. 9); therefore AK is equal to AE, that is, the part equal to the whole (Fig. 1), or the whole equal to the part (Fig. 2): which is impossible (Ax. 9). Therefore the diagonal of BD through A cannot fall otherwise than on AF; that is, AF, AC are in the same straight line. Which was to be proved.

PROP. XXVII. THEOR.

Of all parallelograms applied to the same straight line and deficient by parallelograms, which are each of them similar and similarly situated to the same parallelogram described on the half of the line, that which is described on half the line, and consequently equal to its defect, shall be the greatest.

OBS. The language used in this and the three next propⁿˢ will appear intricate, unless the following explanations be attended to.

When a parallelogram has one of its sides coinciding with a straight line, it is said to be "applied" to that straight line (cf. Bk. i. Propⁿˢ 44, 45). Thus the parallelogram ACDE is "applied to the straight line AB;" i.e. the side AC coincides with AB, B and C being the same point.

When a parallelogram has one of its sides falling on a straight line, with one of whose extremities an extremity of the side coincides, the parallelogram may still be said to be applied to the

straight line without ambiguity, provided we state in addition, that the two other extremities do not coincide. How this is done will be seen in the annexed example.

Ex. Let the side AC of the parallelogram ACDE fall on the straight line AB; the extremity A coinciding with one of AB, and the other C falling either beyond AB in the straight line ABC (Fig. 1), or in AB (Fig. 2). Through B draw BG parallel to AE or CD,

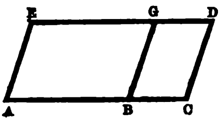


Fig. 1.

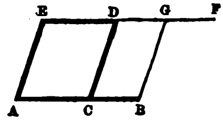


Fig. 2.

cutting ED (Fig. 1), or ED produced to F (Fig. 2) in G. By this const^{n} EB, GC are parallelograms, and applied to AB, BC respectively. And:—

(1) in Fig. 1, EB and GC together make up EC; that is, the parallelogram EC exceeds the parallelogram EB applied to AB by the parallelogram GC applied to BC. Which is briefly expressed by saying that the parallelogram ACDE "is applied to the straight line AB, and exceeding by the parallelogram GC;" and GC is called the "excess" of ACDE;

(2) in Fig. 2, EC and GC together make up EB; that is, the parallelogram EC is deficient from the parallelogram EB applied to AB by the parallelogram GC applied to CB. Which is briefly expressed by saying that the parallelogram ACDE "is applied to the straight line AB, and deficient by the parallelogram GC;" and GC is called the "defect" of ACDE.

Let AB be a straight line bisected in C, and on the half CB let the parallelogram CE be described. Then of all the parallelograms applied to AB, and deficient by parallelograms which are each of them similar and similarly situated to DB, the parallelogram ACDM, whose defect is the parallelogram DB, shall be the greatest.

Let AGFK be any other parallelogram applied to AB, and deficient by the parallelogram FKBH, which is similar and similarly situated to

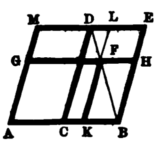


Fig. 1.

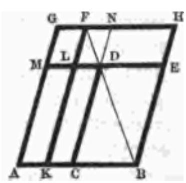


Fig. 2.

DB. There are two cases, according as AK is greater than AC (Fig. 1), or less than AC (Fig. 2).

I. Let AK be greater than AC (Fig. 1).

Since the parallelogram FKBH is by hyp[s] similar and similarly situated to the parallelogram DCBE, the sides BK, BH of the one fall along the homologous sides BC, BE of the other; and therefore their diagonals through the common angular point B are in the same straight line (vi. 26). Draw this straight line DFB, and produce KF to cut DE in L.

Because parallelograms on equal bases and between the same parallels are equal (i. 36), AD is equal to CE, and CG to CH. Again, since DF, HK are parallelograms about the diagonal DB of the parallelogram DB, the complements CF, FE are equal (i. 43); to each of these equals add KH: then the whole CH is equal (Ax. 2) to the whole KE. But CH is equal to GC; and things that are equal to the same thing are equal to one another (Ax. 1): therefore CG is equal to KE. To each of these equals add CF: then the whole AF is equal to the gnomon CHL, and therefore less (Ax. 9) than CE. And CE is equal to AD; therefore AD is greater than AF.

II. Let AK be less than AC (Fig. 2).

Since the parallelogram DCBE is by hyp[s] similar and similarly situated to the parallelogram FKBH, the sides BC, BE of the one fall on the homologous sides BK, BH of the other; and therefore their diagonals through the common angular point B are in the same straight line (vi. 26). Draw this straight line BDF; let DM cut FK in L; and produce CD to cut FH in N.

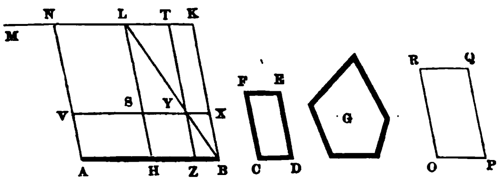
By const[n] AN, CH are parallelograms; and the opposite sides of parallelograms are equal (i. 34): therefore NG is equal to AC, and HN to BC. But AC is by hyp[s] equal to CB; therefore HN is equal to NG. Now the parallelograms DH, DG are on equal bases HN, NG and between the same parallels, therefore DH is equal (i. 36) to DG, and therefore greater (Ax. 9) than LG; but since LN, CE are parallelograms about the diagonal FB of the parallelogram KH, the complements DH, DK are equal: therefore DK is greater than LG. To each of these unequals add AL; then the whole AD is greater (Ax. 4) than the whole AF.

Hence in both cases the parallelogram AD is greater than the parallelogram AGFK; and therefore of all the parallelograms applied to AB, and deficient by parallelograms which are each of them similar and similarly situated to DB, the parallelogram AD is the greatest. Which was to be proved.

PROP. XXVIII. PROB.

To a given straight line to apply a parallelogram, deficient by a parallelogram similar to a given parallelogram, and equal to a given polygon, which is not greater (vi. 27) **than the parallelogram described on the half of the line and similar to the given parallelogram.**

Let AB be the given straight line; CDEF the given parallelogram; G the given polygon, which is not greater than the parallelogram described on half of AB, similar to FD. It is required to apply to AB a parallelogram, deficient by a parallelogram similar to FD, and equal to G.



Bisect (i. 10) AB in H; on HB describe (vi. 18) the parallelogram HBKL similar to FD, and similarly situated with it, viz. having the side HB homologous to the side CD, and HL, LK, KB to CF, FE, ED. Produce KL to M; through A draw (i. 31) AN parallel to HL or BK cutting KM in N; then by const^n AL will be a parallelogram, and since AH is equal to HB, it is equal to HK, and similar to it. Now since by hyp^s G is not greater than HK, HK must be either equal to or greater than G. If HK be equal to G; then, since AL is equal to HK, and things that are equal to the same thing are equal to one another (Ax. 1), AL is also equal to G, and the thing required is already done: for to the given straight line AB has been

applied the parallelogram AL, deficient by the parallelogram HK similar to the given parallelogram FD, and equal to the given polygon G. But if HK be greater than G: describe (vi. 25) the parallelogram OPQR, equal to the excess of HK above G, and similar to FD, having OP, PQ, QR, RS homologous to CD, DE, EF, FG. Then since HK is similar to FD; and polygons that are each of them similar to a third are similar (vi. 20): HK is similar to OQ, and HB, BK, KL, LH are homologous to OP, PQ, QR, RO. Now by const^n OQ is equal to the excess of HK above G; that is, HK is equal to OQ together with G; therefore OQ is less (Ax. 9) than HK: hence its sides OR, RQ are less (vi. 20. Cor. A) than the homologous sides HL, LK. From LH, LK then cut off (i. 3) LS, LT equal to RO, RQ. Through S draw VSX parallel to NK or AB, cutting AN in V and KB in X; and through T draw TYZ parallel to AN or KB, cutting VX in Y and AB in Z. Then VYZA, which by const^n is a parallelogram, shall be the parallelogram required.

The parallelograms ST, OQ have their sides equal by const^n, and the angle TLS is equal to the angle ORQ; therefore they are equal in every respect, and similar (vi. Def. 1. Obs. 4); and HK is similar to OQ: therefore ST is similar to HK, and LS, LT fall on the homologous sides LH, LK. Therefore their diagonals through L are in the same straight line (vi. 26); draw this straight line LYB.

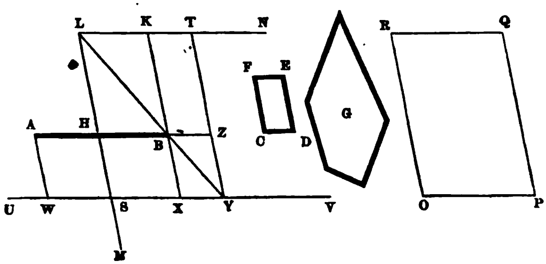
By const^n VH, HX are parallelograms, and they are on equal bases AH, HB and between the same parallels: therefore VH is equal (i. 36) to HX. Again since ST, ZX are parallelograms about the diagonal LB of the parallelogram HK, XZ is similar (vi. 24) to HK, and therefore to CE; and also the complements HY, YK are equal (i. 43); to each of these equals add XZ: then the whole TB is equal (Ax. 2) to the whole HX. But VH is equal to HX; and things that are equal to the same thing are equal to one another: therefore TB is equal to VH. To each of these equals add HY: then the whole VZ is equal to the gnomon HXT. Again since HK is equal to OQ together with G, and ST is equal to OQ, therefore, taking away equals from equals, the remaining gnomon HXT is equal (Ax. 3) to the remainder G; but it has been shewn that VZ is equal to the gnomon HXT, and things that are equal to the same

thing are equal to one another: therefore VZ is equal to G. Hence to the given straight line AB has been applied the parallelogram AVYZ, deficient by a parallelogram YZBX similar to the given parallelogram CDEF, and equal to the given polygon G. Which was to be done.

PROP. XXIX. PROB.

To a given straight line to apply a parallelogram, exceeding by a parallelogram similar to a given parallelogram, and equal to a given polygon.

Let AB be the given straight line, CDEF the given parallelogram, and G the given polygon. It is required to apply to AB a parallelogram, deficient by a parallelogram similar to FD, and equal to G.



Bisect (i. 10) AB in H; on HB describe (vi. 18) the parallelogram HBKL similar to FD, and similarly situated with it, viz. having the side HB homologous to the side CD, and HL, LK, KB to CF, FE, ED; and describe (vi. 25) the parallelogram OPQR, equal to HK and G together, and similar to CDEF, having OP, PQ, QR, RS homologous to CD, DE, EF, FC. Then since HK is similar to FD, and polygons that are each of them similar to a third are similar (vi. 20), HK is similar to OQ, and HB, BK, KL, LH are homologous to OP, PQ, QR, RO. Now by const^{n} OQ is equal to HK and G together; therefore OQ is greater (Ax. 9) than HK, and hence its sides OR, RQ are greater (vi. 20. Cor. A) than the homologous sides HL, LK. Produce then LH, LK to M and N, and from them cut off

(i. 3) LS, LT equal to RO, RQ. Through S draw (i. 31) USV parallel to LN or AB; through A, T draw AW, TY parallel to HL or BK, cutting UV in W, Y; and produce KB, HB to cut WY, YT in X and Z. Then AWYZ, which is a parallelogram by const^n shall be the parallelogram required.

The parallelograms ST, OQ have their sides equal by const^n and the angles SLT, ORQ equal; therefore they are equal in every respect, and similar (vi. Def. 1. Obs. 4); and HK is similar to OQ: therefore HK is similar to ST, and LH, LK fall on the homologous sides LS, LT. Therefore their diagonals through L are in the same straight line (vi. 26); draw this straight line LBY.

Because the parallelograms AS, HX are on equal bases AH, HB, and between the same parallels AB, WX, therefore AS is equal (i. 36) to HX. Again, since HK, XZ are parallelograms about the diagonal LY of the parallelogram ST, XZ is similar (vi. 24) to ST, and therefore to FD (vi. 21); and also the complements HX, KZ are equal (i. 43). But AS was shewn to be equal to HX; and things that are equal to the same thing are equal to one another (Ax. 1): therefore AS is equal to KZ. To each of these equals add SZ; therefore the whole AY is equal (Ax. 2) to the gnomon SZK. Now since HK and G are together equal to OQ, that is, to ST, from each of these equals take away HK; then the remainder G is equal (Ax. 3) to the remaining gnomon SZK; but it has been shewn that AY is equal to the gnomon SZK, and things that are equal to the same thing are equal to one another: therefore AY is equal to G. Hence to the given straight line AB has been applied the parallelogram AWYZ, exceeding by the parallelogram BXYZ similar to the given parallelogram CDEF, and equal to the given polygon G. Which was to be done.

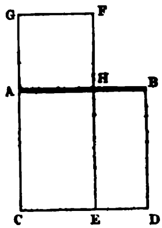
PROP. XXX. PROB.

To cut a given line in extreme and mean ratio.

Let AB be the given straight line. It is required to cut it in extreme and mean ratio.

There are two methods of solving this problem, one by

the aid of the preceding propⁿ, and the other by Bk. II. Prop. XI.



I. On AB describe (i. 46) the square ACDB; and supposing A to be that extremity of AB adjacent to which the greater part is required to be, to CA apply (vi. 29) the parallelogram CEFG exceeding by the parallelogram AHFG similar to the square AD, and equal to the square AD. Since GH is similar to AD, it will be a square; and as in Bk. II. Prop. XI. H will be a point in AB. Then AB shall be cut in H as required.

By constⁿ FC is equal to AD; from each of these equals take away the common part AE: then the remainder GH is equal (Ax. 3) to the remainder HD. Now GH is a square, and as in Bk. II. Prop. XI. HD is a rectangle; hence the two parallelograms GH, HD, having the angles AHF, BHE right angles, and therefore equal (Ax. 11), are equal. Therefore their sides about these angles are reciprocally proportional (vi. 14), that is, EH is to HF as AH to HB. But since the opposite sides of parallelograms are equal (i. 34), EH is equal to BD, that is, to BA, for they are sides of the square AD (Def. 30); and HF is equal to AH, since they are sides of the square GH: therefore AB is to AH as AH to HB, and AH is greater (v. A) than HB, because AB is greater than AH (Ax. 9). Hence AB is divided in H so that the whole AB is to the greater part AH as AH is to the less part HB; that is, the given straight line AB has been cut in extreme and mean ratio (vi. Def. 3) in D. Which was to be done.

II. Divide (ii. 11) AB in H, so that the rectangle AB, BH is equal to the square of AH. Then AB shall be cut in H as required.

By constⁿ the rectangle contained by the extremes AB, BH of the three straight lines AB, AH, BH is equal to the square of the mean AH; therefore these three straight lines are proportional (vi. 17), that is, AB is to AH as AH to HB. Hence, as before, the given straight line AB has

been cut in extreme and mean ratio in H. Which was to be done.

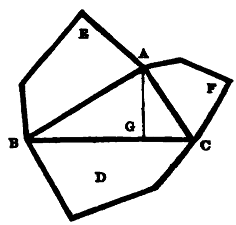
PROP. XXXI. THEOR.

In any right-angled triangle, a polygon described on the side opposite to the right angle shall be equal to the polygons which are described on the sides including the right angle similar to it and similarly situated with it.

Let ABC be a right-angled triangle, having the right angle BAC; let D be a polygon described on the side BC opposite the right angle BAC; and let E, F be polygons described on the sides AB, AC, including the right angle, similar to D, and similarly situated with it, viz. having their sides AB, AC homologous to the side BC. Then D shall be equal to E and F.

From A draw (i. 12) AG perpendicular to BC.

Because BC is divided into the two parts BG, GC by the perpendicular AG drawn from the right angle A; therefore (vi. 8. Cor.) CB is to BA as BA to BG, and CB is to CA as CA to CG. But if three straight lines are proportional, the first is to the third as the polygon on the first to the similar and similarly described polygon on the second

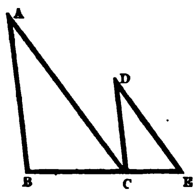


(vi. 20. Cor); therefore CB is to BG as D to E, and CB is to CG as D to F. Hence, invertendo (v. B), BG is to BC as E to D, and GC is to BC as F to D; therefore (v. 24) BG and GC together are to BC as E and F together are to D. Now BG and GC together are equal to BC: therefore (v. A) E and F together are equal to D; that is, the polygon D described on the side BC opposite to the right angle is equal to the polygons E, F, described on the sides AB, AC including the right angle, similar to D and similarly situated with it. Which was to be proved.

PROP. XXXII. THEOR.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at a common angular point so as to have their homologous sides in the proportion parallel: then their third sides shall be in one straight line.

Let the two triangles ABC, DCE have the two sides BA, AC proportional to the two sides CD, DE, viz. BA to AC as CD to DE; and let them be joined at the common angular point C, so that AB is parallel to DC and AC to DE. Then the third sides BC, CE shall be in one straight line.



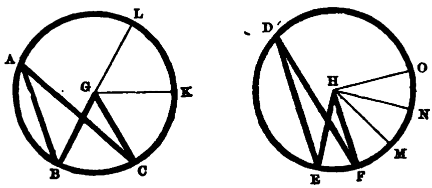
Because AC cuts AB, CD, which are parallels by hyp[s], in A, C, the alternate angles BAC, ACD are equal (i. 29); and because CD cuts AC, DE which are parallels by hyp[s] in C, D, the alternate angles ACD, CDE are equal. Now things that are equal to the same thing are equal to one another (Ax. 1); therefore the angles BAC, CDE are equal. And by hyp[s] BA is to AC as CD to DE; therefore the two triangles BAC, CDE have the angle BAC equal to the angle CDE, and the sides about this pair of equal angles proportional. Therefore these two triangles are similar (vi. 6); and hence the angle ABC is equal to the angle DCE. Now the angle BAC was shewn to be equal to the angle ACD; therefore, adding equals to equals, the two angles CBA, BAC are equal (Ax. 2) to the whole angle ACE. To each of these equals add the angle ACB; then the three angles CBA, BAC, ACB are equal to the two ACE, ACB. But the three angles CBA, BAC, ACB of the triangle ABC are equal (i. 32) to two right angles; and things that are equal to the same thing are equal to one another: therefore the two angles ACE, ACB are equal to two right angles. That is, the two straight lines CE, CB on opposite sides of CA make with it at the point C the adjacent angles ACE, ACB equal to two right angles; therefore BC, CE are on one straight line (i. 14). Which was to be proved.

PROP. XXXIII. THEOR.

In equal circles or in the same circle angles, whether they be at the centres or circumferences, shall have the same ratio to one another which the arcs on which they stand have. And in equal circles or in the same circle sectors shall have the same ratio to one another which the arcs on which they stand have.

Let ABC, DEF be equal circles, of which the centres are G, H respectively; and:—

I. Let the angles BGC, EHF at the centres, and the angles BAC, EDF at the circumferences, stand on the arcs BC, EF. Then the arc BC shall be to the arc EF as the angle BGC is to the angle EHF, and as the angle BAC to the angle EDF.



In the circle ABC take any number of arcs BC, CK, KL each equal to BC; and in the circle DEF any number FM, MN, NO, each equal to EF. Join GK, GL; HM, HN, HO.

Because the arcs BC, CK, KL are equal by const^{n}, and in the same circle the angles at the centre which stand on equal arcs are equal (iii. 27): therefore the angles BGC, CGK, KGL are all equal, and hence whatever multiple the arc BL is of the arc BC the same multiple the angle BGL is of the angle BGC; that is, the arc BL and the angle BGL are equimultiples of the arc BC and the angle BGC. By like reasoning, the arc EO and the angle EHO are equimultiples of the arc EF and the angle EHF. Also, since in equal circles the angles which stand on equal arcs are equal (iii. 27), if the arc BL be equal to the arc EO, the angle BGL will be equal to the angle EHO; if the arc BL be greater than the arc EO, the angle BGL will be greater than the angle EHO; and if less, less. Now since there

are four magnitudes, viz. the arc BC, the arc EF, the angle BGC, the angle EHF, and there have been taken of the first, the arc BC, and the third, the angle BGC, any equimultiples whatever, the arc BL and the angle BGL, and of the second, the arc EF, and the fourth, the angle EHF, any equimultiples whatever, the arc EO and the angle EHO; and since it has been shewn that the multiple of the third, the angle BGL, is greater than, equal to, or less than that of the fourth, the angle EHO, according as the multiple of the first, the arc BL, is greater than, equal to, or less than that of the second, the arc EO: therefore by the def^n of proportion (v. Def. 5), the arc BC is to the arc EF as the angle BGC is to the angle EHF:

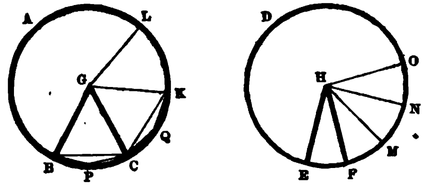
Again, because the angle BAC at the circumference and the angle BGC at the centre of the circle ABC have the same arc BC for their base, the angle BGC is double (iii. 20) of the angle BAC. In like manner the angle EHF is double of the angle EDF; and magnitudes have to one another the same ratio which their equimultiples have (v. 15): therefore the angle BAC is to the angle EDF as the angle BGC to the angle EHF. But it has been shewn that the arc BC is to the arc EF as the angle BGC to the angle EHF; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore the arc BC is to the arc EF as the angle BAC to the angle EDF:

Hence the arc BC is to the arc EF both as the angle BGC is to the angle EHF, and as the angle BAC is to the angle EDF. Had the arcs and angles been all in the same circle ABC instead of in equal circles, the proof would have been exactly the same, H in this case coinciding with G, and D, E, F, M, N, O being points in the circumference ABC. Which was to be proved.

II. Let the sectors BGC, EHF stand on the arcs BC, EF. Then the arc BC shall be to the arc EF as the sector BGC is to the sector EHF.

Construct as in Part I. Also join BC, CK; in the arcs BC, CK take any points P, Q; and join BP, PC, CQ, QK.

By the def^n of a circle BG, GC, GK are all equal, and for the same reason as in Part I. of the prop^n the angles BGC, CGK are equal; therefore the two triangles BGC, CGK

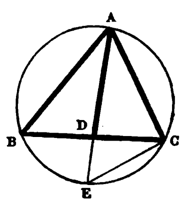


have the two sides BG, GC respectively equal to the two sides CG, GK, and the included angle BGC equal to the included angle CGK. Therefore these two triangles are equal in every respect (i. 4); and hence the base BC is equal to the base CK, and the triangle BGC to the triangle CGK. Again, the arc BPC is equal to the arc CQK by const^n; hence, taking away each of these equals from the same magnitude, viz. the whole circumference ABC, the remaining arc BAC is equal (Ax. 3) to the remaining arc CAK. But in the same circle the angles at the circumference which stand on equal arcs are equal (iii. 27); therefore the angle BPC is equal to the angle CQK. Hence the segments BPC, CQK are similar (iii. Def. 11); and they stand on equal straight lines BC, CK: therefore they are equal (iii. 24) to one another. And the triangle BGC was proved to be equal to the triangle EHF; therefore, adding equals to equals, the whole sector BGC is equal (Ax. 2) to the whole sector CGK. In like manner it may be shewn that the sector KGL is equal to the sector BGC, and that the sectors BGC, CGK, KGL are all equal; and hence whatever multiple the arc BL is of the arc BC, the same multiple is the sector BGL of the sector BGC; that is, the arc BL and the sector BGL are equimultiples of the arc BC and the sector BGC. By like reasoning the arc EO and the sector EHO are equimultiples of the arc EF and the sector EHF. Also if the arc BL be equal to the arc EN, it may be shewn that the sector BGL will be equal to the EHO by const^n and proof similar to that used above to prove the sectors BGC, CGK equal, the equal circles ABC, EHN having equal radii and circumferences (iii. Def. 1), and being now substituted for the same circle ABC; if the arc BL be greater than the arc EN, the sector BGL will be greater than the sector EHO; and if less, less. Now

since there are four magnitudes, viz. the arc BC, the arc EF, the sector BGC, the sector EHF, and there have been taken of the first, the arc BC, and the third, the sector BGC, any equimultiples whatever, the arc BL and the sector BGL, and of the second, the arc EF, and the fourth, the sector EHF, any equimultiples whatever, the arc EO and the sector EHO; and since it has been shewn that the multiple of the third, the sector BGL, is greater than, equal to, or less than that of the fourth, the sector EHO, according as the multiple of the first, the arc BL, is greater than, equal to, or less than that of the second, the arc EO: therefore by the def^n of proportion (v. Def. 5) the arc BC is to the arc EF as the sector BGC is to the sector EHF. Had the arcs and sectors been all in the same circle ABC instead of in equal circles, the proof would have been exactly the same, H in this case coinciding with G, and D, E, F, M, N, O being points in the circumference ABC. Which was to be proved.

PROP. B. THEOR.

If an angle of a triangle be bisected by a straight line cutting the opposite side: then the rectangle contained by the two other sides of the triangle shall be equal to that contained by the two parts into which this side is divided together with the square of the bisecting line.



Let ABC be a triangle, and let one of its angles BAC be bisected by AD, cutting the opposite side BC in D. Then the rectangle BA, AC shall be equal to the rectangle BD, DC together with the square of AD.

About the triangle ABC circumscribe (iv. 5) the circle ABC; produce AD to meet the circumference in E; and join EC.

By hyp^s the angle BAD is equal to the angle CAD, and the angle ABC is equal (iii. 21) to the angle AEC, because they are in the same segment ABEC; hence the

triangles ABD, AEC have the two angles ABD, BAD respectively equal to the two angles AEC, EAC, and consequently (i. 32. Cor. A) the third angle BDA to the third angle ACE; therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence BA is to AD as EA to AC. But if four straight lines be proportional, the rectangle contained by the two extremes is equal (vi. 16) to that contained by the two means: therefore the rectangle BA, AC is equal to the rectangle EA, AD. Now, since AE is divided into two parts in D, the rectangle EA, AD is equal (ii. 3) to the rectangle ED, DA together with the square of AD; and because AE, BC are straight lines in the circle, cutting one another in D, the rectangle ED, DA is equal (iii. 33) to the rectangle BD, DC: therefore the rectangle EA, AD is equal to the rectangle BD, DC together with the square of AD. But it was shewn that the rectangle BA, AC is equal to the rectangle EA, AD; and things that are equal to the same thing are equal to one another (Ax. 1): therefore the rectangle BA, AC is equal to the rectangle BD, DC together with the square of AD. Which was to be proved.

PROP. C. THEOR.

If from any angular point of a triangle a straight line be drawn perpendicular to the opposite side, or the opposite side produced: then the rectangle contained by the two other sides of the triangle shall be equal to that contained by the perpendicular and the diameter of the circle circumscribed about the triangle.

Let ABC be a triangle, and A one of its angular points: from A let AE be drawn perpendicular to the opposite side BC (Fig. 1), or BC produced to D (Fig. 2); and let ABF be the circle circumscribed about the triangle ABC. Then the rectangle BA, AC shall be equal to that contained by AE and the diameter of the circle ABF.

Find (iii. 1) the centre of the circle ABC, and through A draw the diameter AG; join GC.

The angle ACG in the semicircle ACG is a right angle (iii. 31), and AEB is a right angle by hyp[s]; and all right

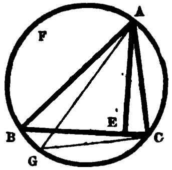


Fig. 1.

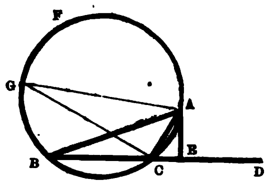


Fig. 2.

angles are equal (Ax. 11): therefore the angle ACG is equal to the angle AEB. Also the angle ABC is equal (iii. 21) to the angle AGC, because they are in the same segment ABGC; hence the two triangles ABE, AGC have the two angles AEB, ABE respectively equal to the two angles ACG, AGC, and consequently (i. 32. Cor. A) the third angle BAE equal to the third angle GAC: therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence BA is to AE as GA to AC. But if four straight lines be proportional, the rectangle contained by the two extremes is equal (vi. 16) to that contained by the two means; therefore the rectangle BA, AC is equal to the rectangle AE, GA, that is, to the rectangle contained by AE and the diameter of the circumscribed circle ABF. Which was to be proved.

PROP. D.

The rectangle contained by the two diagonals of any quadrilateral inscribed in a circle shall be equal to the two rectangles contained by the two pairs of its opposite sides.

Let ABCD be a quadrilateral figure inscribed in the circle ABC; AC, BD its two diagonals, that is, the straight lines joining the two pairs of opposite angular points. Then the rectangle AC, BD shall be equal to the rectangle AB, CD together with the rectangle AD, BC.

The two angles ABD, DBC are either equal to one another, or not. If they are equal (Fig. 1), let F be the point where BD cuts AC. If they are unequal (Fig. 2),

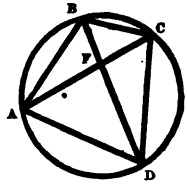


Fig. 1.

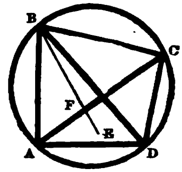


Fig. 2.

let ABD be the one which is greater than the other CBD; and at the point B in the straight line AB make (i. 23) the angle ABE equal to the angle DBC, and let F be the point where BE cuts AC.

In Fig. 1, the angle ABD is equal to the angle CBF. And in Fig. 2, since by constⁿ the angle ABF is equal to the angle CBD, to each of these equals add the angle DBF: then the whole angle ABD is equal (Ax. 2) to the whole angle CBF. Hence in both figures the angle ABD is equal to the angle CBF; and the angle BCA is equal to the angle BDA, because they are in the same segment BCDA (iii. 21); therefore the two triangles ABD, CBF have the two angles ABD, ADB respectively equal to the two angles CBF, BCF, and consequently (i. 32. Cor. A) the third angle BAD to the third angle CFB; that is, these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence BC is to CF as BD to DA. But if four straight lines be proportional, the rectangle contained by the two extremes is equal (vi. 16) to that contained by the two means: therefore the rectangle BC, AD is equal to the rectangle BD, CF. Again in both figures the angle ABF is equal to the angle CBD, and the angle BAC to the angle BDC, because they are in the same segment of the circle BADC; hence the two triangles BAF, BCD have the two angles ABF, BAF respectively equal to the two angles DBC, CDB, and consequently the third angle AFB to the third angle BCD; that is, these two triangles are equiangular to one another. Therefore they are similar; and hence BA is to AF as BD to DC. Therefore for the same reason as before, the rectangle BA, DC is equal to the rectangle BD, AF. Now it has been shewn that the rectangle BC, AD is equal to the rectangle BD, CF: hence,

adding equals to equals, the rectangles BC, AD and AB, DC are together equal to the rectangles BD, CF and BD, AF. But since BD is undivided, and CA divided into two parts in F, the rectangle BD, CA is equal (ii. I) to the rectangles BD, CF and BD, AF; and things that are equal to the same thing are equal to one another (Ax. I): therefore the rectangle AC, BD is equal to the two rectangles AB, DC and AD, BC. Which was to be proved.

THE

ELEMENTS OF EUCLID.

BOOK XI.

DEFINITIONS.

I.

A SOLID is that which has length, breadth, and thickness.

II.

The extremities of a solid are surfaces.

OBS. Hence a solid figure (Bk. i. Def. 14) is enclosed by one or more boundaries (Bk. i. Def. 13), each of which is a surface.

III.

A straight line is defined to be perpendicular or at right angles to a plane, when it is at right angles to every straight line that is drawn to meet it in that plane.

IV.

A plane is defined to be perpendicular or at right angles to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

OBS. The common section of two surfaces which cut one another is a line; and it will be proved in Bk. xi. Prop. 3, that in the case of two planes cutting one another, this line is a straight line.

V.

When a straight line meets a plane and is neither in the same plane with it nor at right angles to it, it is said to be obliquely inclined to the plane; and its inclination is the acute angle included by the straight line and another straight line, which is drawn from the point where the first line meets the plane to the point in which a perpendicular to the plane, drawn from any point of the first line, meets the plane.

VI.

When a plane cutting another plane is not perpendicular to it, the two planes are said to be obliquely inclined to one another; and their inclination is the acute angle included by two straight lines drawn each from any point in the common section of the two planes at right angles to it, one in one plane and the other in the other plane.

VII.

Two planes are said to have the same or a like inclination to one another which two other planes have, when the angles of inclination are equal to one another.

VIII.

Parallel planes are such as being produced ever so far every way never meet.

IX.

A solid angle is that which is constituted by the meeting together at a point of more than two plane angles, each of which is in a plane different from all the others, and is contiguous to two of them, i. e. each of the two straight lines which include it is one of the straight lines which include two of the other plane angles.

Obs. 1. A solid angle may also be regarded as formed by the meeting together of more than two planes at a point, each of which is dif-

ferent from all the others, and is intersected by two of them; for the common sections of each with the two contiguous planes (which are straight lines by Bk. xi. Prop. 3) form one of the plane angles, which in the def^n constitute the solid angle.

OBS. 2. If the plane angles which constitute one solid angle be respectively equal to those constituting another, and the inclination of planes of each pair of contiguous angles in the one is equal to that of the planes of the pair of contiguous equal angles in the other: then these two solid angles shall be equal to one another.

For if one solid angle be applied to the other, so that a plane angle of the one may coincide with the equal plane angle of the other, the two solid angles falling on the same side of it; then the two planes contiguous to the plane angle of the one will fall on the two planes contiguous to the equal plane angle of the other, since by hyp^s the inclinations of each pair of contiguous angles are equal in each; and therefore the two plane angles contiguous to the plane angle of the one will coincide with the two contiguous to that of the other, since by hyp^s the plane angles are equal. Similarly it may be shewn that each of the other plane angles of the one coincides with each of those of the other. Hence the two solid angles coincide; and magnitudes that coincide are equal (Ax. 7): therefore the two solid angles are equal to one another. Which was to be proved.

X.

When the surfaces which bound a solid figure are all planes, the solid figure is called a polyhedron.

OBS. 1. Since the common section of two planes cutting one another is a straight line (Bk. xi. Prop. 3), each of the bounding planes of the polyhedron will be cut by the other bounding planes in straight lines, and will therefore be a plane rectilinear figure or polygon, of which these straight lines are the sides, and at each of whose angular points more than two bounding planes cut one another. Then:—

(1) the straight lines, which are the common sections of pairs of the bounding planes, and are sides common to two of the polygons, are called the edges of the polyhedron;

(2) the bounding polygons are called the faces of the polyhedron;

(3) the solid angles at the points where more than two of the bounding planes meet, which are constituted by the plane angles of the corresponding polygons, are called the solid angles of the polyhedron.

OBS. 2. When the faces of a polyhedron are all equal regular polygons, it is called a regular polyhedron.

XI.

One polyhedron is defined to be similar to another polyhedron having the same number of solid angles, and the same number of faces constituting each solid angle, when each solid angle of the one is equal to a solid angle of the other, and the faces constituting each solid angle of the one are respectively similar to the faces constituting the equal solid angle of the other.

XII.

A pyramid is a polyhedron, bounded by a polygon, and a set of triangles which have the sides of the polygon for their bases, and meet in a common angular point without the plane of the polygon.

XIII.

A prism is a polyhedron, two of the faces of which are equal, similar and parallel to one another, and the rest parallelograms.

XIV.

A sphere is the solid figure generated by the revolution of a semicircle about its diameter, which remains fixed.

XV.

The axis of a sphere is a fixed straight line about which the generating semicircle revolves.

XVI.

The centre of a sphere is the centre of the generating semicircle.

XVII.

A diameter of a sphere is a straight line which passes through the centre, and which is terminated both ways by the surface of the sphere.

XVIII.

A right cone is the solid figure generated by the revolution of a right-angled triangle about one of the sides including the right angle, which remains fixed.

The cone is called a right-angled, an obtuse-angled, or an acute-angled cone, according as the fixed side of the generating triangle is equal to, less than, or greater than the other side including the right angle.

OBS. Unless the contrary be expressly stated, whenever a cone is spoken of, a right cone is to be understood.

XIX.

The axis of a cone is the fixed straight line about which the generating triangle revolves.

XX.

The base of a cone is the circle swept out by that side of the generating triangle including the right angle, which revolves.

XXI.

A cylinder is the solid figure generated by the revolution of a rectangle about one of its sides, which remains fixed.

XXII.

The axis of a cylinder is the fixed straight line about which the generating rectangle revolves.

XXIII.

The bases of a cylinder are the two circles swept out by the opposite sides of the generating rectangle, which revolve.

XXIV.

Cones and cylinders are defined to be similar when they have their axes and the radii of their bases proportional.

XXV.

A cube is a polyhedron bounded by six equal squares.

XXVI.

A regular tetrahedron is a polyhedron bounded by four equal equilateral triangles.

XXVII.

A regular octahedron is a polyhedron bounded by eight equal equilateral triangles.

XXVIII.

A regular dodecahedron is a polyhedron bounded by twelve equal regular pentagons.

XXIX.

A regular icosahedron is a polyhedron bounded by twenty equal equilateral triangles.

OBS. The polyhedrons defined in the five preceding defns are regular polyhedrons (Bk. xi. Def. 10. Obs. 2); and it can be proved that these five are the only regular polyhedrons which exist. Unless the contrary be expressly stated, when a tetrahedron, octahedron, dodecahedron, or icosahedron is spoken of, a regular one is to be understood.

A.

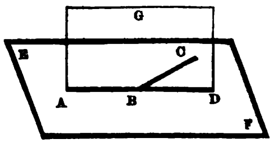
A parallelopiped is a polyhedron bounded by six parallelograms, each opposite pair of which are equal and parallel.

PROPOSITIONS.

PROP. I. THEOR.

One part of a straight line cannot be in a plane and another part without the plane.

For if there can, let. if possible, one part AB of a straight line ABC be in the plane EF, and another part BC without the plane. Then since the straight line AB is in the plane EF, it can be produced to any length required in a straight line in this plane (Post. 2); let it be produced to a point D in the plane EF. Also take some plane passing through the straight line AD; and let it revolve about AD, remaining fixed, until it pass through the point C, and let it then have arrived into the position AGD, C being a point, in the plane AGD.



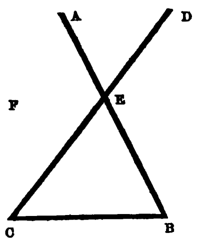
Then because the points B, C are both in the plane AGD, the straight line BC is in this plane, by the def^n of a plane (Def. 7). Hence there are two straight lines ABC, ABD in the same plane AGD which have a common segment AB; which is impossible (i. 11. Cor.). Therefore one part of a straight line cannot be in a plane, and another part without the plane. Which was to be proved.

PROP. II. THEOR.

Two straight lines which cut one another shall be in one plane. And three straight lines, each of which cuts the other two, shall be in one plane.

I. Let the two straight lines AB, CD cut one another in E. Then AB, CD shall be in one plane.

Take some plane passing through AB; and let it revolve about AB, remaining fixed, until it pass through the point C, and let it then have arrived into the position BAF, C being a point in the plane BAF.



Then because the points E, C are both in the plane BAF, the straight line EC which joins them is in this plane by the def[n] of a plane (Def. 7). Therefore AB, EC are in the plane BAF; but CD is in the same plane BAF that CE is in, since one part of a straight line cannot be in a plane, and another part without it (xi. 1): therefore AB, CD are both in one plane ABF. Which was to be proved.

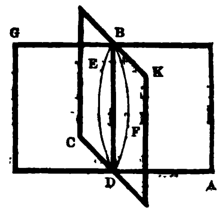
II. Let EC, CB, BE be three straight lines, each of which cuts the other two, the points of intersection being E, C, B. Then they shall all three be in one plane.

For it may be shewn as in Part I. of the prop[n] that the straight line EC is in a plane BEF which passes through BE, and that the straight line BC is in the plane BEF. Therefore EB, BC, CE are all three in one plane BEF. Which was to be proved.

PROP. III. THEOR.

If two planes cut one another: then their common section shall be a straight line.

Let the two planes AG, CK cut one another, and let their common section be the line DB. Then the line DB shall be a straight line.



For if not: let, if possible, DB be not a straight line. Then since B, D are two points in the plane AG, the straight line which joins them is in this plane (Def. 7), and it does not coincide with BD; let it be BED. Also since B, D are two points in the plane CK, the straight line which joins them is in this plane, and it does not coincide with BD; let it be BFD.

Because BED, BFD are in two different planes, and do not coincide with BD, they are different straight lines, and they have the same points B, D for their extremities; therefore the two straight lines BED, BFD enclose a space: which is impossible (Ax. 10). Therefore BD is a straight line. Which was to be proved.

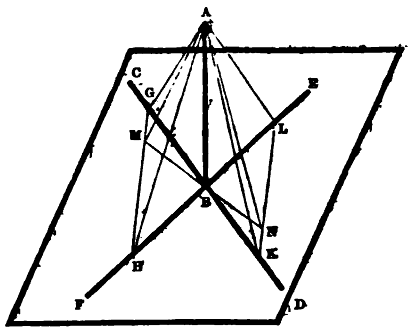
PROP. IV. THEOR.

If a straight line be at right angles to each of two straight lines cutting one another at the point of their intersection: then it shall be perpendicular to the plane, which passes through them, or in which they are.

Let the straight line AB be at right angles to each of the two straight lines CD, EF which cut one another at the point of their intersection B. Then AB shall be perpendicular to the plane (xi. 2) passing through EF, CD.

In BC take any point G, and from BF, BD, BE cut off (i. 3) BH, BK, BL each equal to BG. Join GH, LK; and through B draw in the plane, in which CD, EF are, any straight line MBN, cutting GH in M and LK in N. Lastly, join AG, AM, AH, AK, AN, AL.

By constⁿ the four straight lines BG, BH, BK, BL are equal. And the angles GBH, LBK are equal, since they are opposite vertical angles (i. 15); therefore the two triangles GBH, KBL have the two sides GB, BH respectively equal to the two sides KB, BL and the included angle GBH equal to the included angle KBL. Therefore these two triangles are equal in every respect (i. 4); and hence the



base GH is equal to the base KL, and the angle HGB equal to the angle BKL:

And the angles GBM, KBN are equal, since they are opposite vertical angles; therefore the two triangles GBM, KBN have the two angles MGB, MBG respectively equal to the two angles BKN, NBK, and the sides BG, BK adjacent to the equal angles in each equal. Therefore these two triangles are equal in every respect (i. 26); and hence the side GM is equal to the side KN, and the side MB to the side NB:

Again BG is equal to BK, BA common to the two triangles ABG, ABK, and the angles ABG, ABK are equal (Def. 10), because AB by hyp. is at right angles to CD; therefore these two triangles have the two sides GB, BA respectively equal to the two sides KB, BA and the included angle GBA equal to the included angle KBA. Therefore they are equal in every respect; and hence the base AG is equal to the base AK:

And in like manner from the pair of equal triangles ABH, ABE it may be shewn that the base AH is equal to the base AL:

Also GH was proved to be equal to KL; therefore the two triangles AGH, AKL have the three sides AG, GH, HA respectively equal to the three sides AK, KL, LA. Therefore these two triangles are equal in every respect (i. 8); and hence the angle AGH is equal to the angle AKL:

And AG is equal to AK, and GM to KN; therefore the two triangles AGM, AKN have the two sides AG, GM respectively equal to the two sides AK, KN, and the included angle AGM equal to the included angle AKN. Therefore these two triangles are equal in every respect; and hence the base AM is equal to the base AN:

And lastly, MB was shewn to be equal to BN, and AB is common to the two triangles ABM, ABN: therefore these two triangles have the three sides, AM, MB, BA respectively equal to the three sides AN, NB, BA. Therefore they are equal in every respect; and hence the angle ABM is equal to the angle ABN:

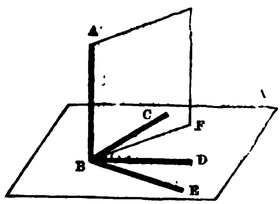
Thus AB standing on MN makes with it the adjacent angles ABM, ABN equal to one another; therefore by the def^n of a right angle each of these angles is a right angle, and AB is at right angles to MBN. In like manner it may be shewn that AB is at right angles to every other straight line drawn meeting it in the plane passing through CD, EF: GL, HK being joined instead of GH, KL, when the straight line lies between BG and BE, and BF and BD, instead of between BC and BF, and EB and BD as in the figure: therefore by the def^n of a straight line being perpendicular to a plane (xi. Def. 3), AB is perpendicular to the plane passing through CD, EF. Which was to be proved.

PROP. V. THEOR.

If a straight line be at right angles to each of three straight lines at the point where they meet: then these three straight lines shall be in one plane.

Let AB be at right angles to each of the three straight lines BC, BD, BE at the point B where they meet. Then BC, BD, BE shall be in one plane.

For if they are not all in one plane; then, since each two must be in one plane because they cut one another (xi. 2), any two will be in one plane, and the third without it. Let, if possible, BD, BE be in one plane, while BC is without this plane through BD, BE; and through AB, BC draw a plane cutting the plane through BD, BE in the common section BF, which is a straight line (xi. 3), and



which cannot coincide with BC, because BC is supposed without the plane through BD, BE.

By the constr BA, BC, BF are all in one plane, viz. that through BA, BC. Now because AB is at right angles to each of the straight lines BD, BE at the point of their intersection B, it is perpendicular to the plane through them (xi. 4); and BF is drawn meeting it in that plane: therefore by the defn of a straight line being perpendicular to a plane (xi. Def. 3), AB is at right angles to BF, and ABF a right angle. But ABC is a right angle by hyps; and all right angles are equal (Ax. 10): therefore the angle ABF is equal to the angle ABC. Now, they are both in one plane, that through BA, BC, so that ABC is a part of the whole ABF; hence the whole ABF is equal to the part ABC: which is impossible (Ax. 9). Therefore no one of the three straight lines BC, BD, BE can be without the plane in which the other two are; that is, the three BC, BD, BE are all in one plane. Which was to be proved.

PROP. VI. THEOR.

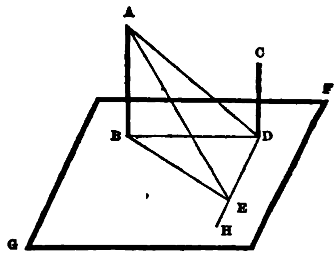
If two straight lines be each of them perpendicular to the same plane: then they shall be parallel to one another.

Let the two straight lines, AB, CD be perpendicular to the same plane FG at the points B, D. Then AB shall be parallel to CD.

Join BD, which by the defn of a plane (Def. 7) is in the plane FG; in this plane draw (i. 11) DH at right angles to BD, and from DH cut off (i. 3) DE equal to AB. Join BE, AE, AD.

Because AB is perpendicular to the plane FG by hyps; and DB, EB are drawn meeting it in that plane: therefore by the defn of a straight line being perpendicular to a

plane (xi. Def. 3), each of the angles ABD, ABE is a right angle. For like reason each of the angles CDB, CDE is a right angle. Now because AB is equal to DE by constⁿ, BD common to the two triangles ABD, BDE, and the right angle



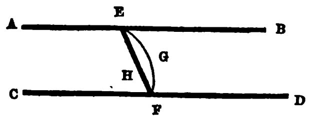
ABD equal to the right angle BDE, since all right angles are equal (Ax. 11); therefore these two triangles have the two sides AB, BD respectively equal to the two sides ED, DB and the included angle ABD equal to the included angle EDB. Therefore they are equal in every respect (i. 4); and hence the base AD is equal to the base BE. And AB is equal to DE, and AE common to the two triangles ABE, ADE; therefore these two triangles have the three sides AB, BE, EA respectively equal to the three sides ED, DA, AE. Therefore they are equal in every respect (i. 8); and hence the angle ABE is equal to the angle EDA. But ABE is a right angle; therefore also EDA is a right angle. And EDB, EDC are both right angles; hence ED is at right angles to each of the three straight lines DB, DA, DC at the point D where they meet. Therefore these three straight lines are all in one plane (xi. 5); and AB is in the plane in which BD, DA are, because each of the three straight lines AB, BD, DA cuts the other two (xi. 2): therefore AB, BD, DC are all in one plane. Now each of the angles ABD, BDC is a right angle; and hence BD cutting the two straight lines AB, CD, which have been proved to be in the same plane with it, in B and D makes the two interior angles ABD, CDB on the same side of BD together equal to two right angles: therefore AB is parallel (i. 28) to CD. Which was to be proved.

PROP. VII. THEOR.

If two straight lines be parallel: then the straight line drawn joining any point in the one with any point in the other shall be in the same plane with the two parallels.

Let AB, CD be two parallel straight lines; E any point in AB; F any point in CD. Then the straight line EF, which joins E and F, shall be in the same plane with AB, CD.

For if not: let it, if possible, be without the plane and have the position EGF, and in the plane in which the parallels AB, CD are draw (Post. 1) the straight line EHF from E to F, which does not coincide with EGF.



Then EGF, EHF, having the same points E, F for their extremities, enclose a space between them; and they are each straight lines; that is, two straight lines enclose a space: which is impossible (Ax. 10). Hence EF cannot be without the plane in which AB, CD are, that is, it is in it. Which was to be proved.

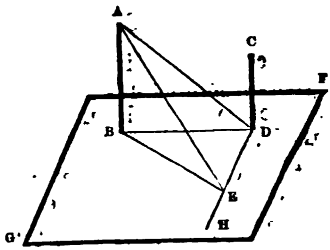
OBS. This propⁿ may be proved at once from the defⁿ of a plane; for since E, F are points in the plane in which the parallels AB, CD are, therefore EF which joins them lies in this plane (Def. 7).

PROP. VIII. THEOR.

If two straight lines be parallel, and one of them be perpendicular to a plane: then the other also shall be perpendicular to that plane.

Let AB, CD be two parallel straight lines, meeting the plane FG in the points B, D; and let one of them AB be perpendicular to the plane. Then the other CD shall also be perpendicular to the plane FG.

Join BD, which by the defⁿ of a plane (Def. 7) is in the plane FG; in this plane draw (i. 11) DH at right angles to BD; and from DH cut off (i. 3) DE equal to AB. Join BE, AE, AD.



Then because AB, CD are parallel by hyp^s, and BD is drawn joining the point B in AB with the point D in CD; BD is in the same plane with AB, CD (xi. 7), or AB, BD, CD are all in one plane. Also because BD cuts the parallels AB, CD, in B, D, the two interior angles ABD, CDB on the same side of it are equal (i. 29) to two right angles. And because AB is by hyp^s perpendicular to the plane FG; and DB, EB are drawn meeting it in that plane: therefore by the def^n of a straight line being perpendicular to a plane (xi. Def. 3), each of the angles ABD, ABE is a right angle. But the angles ABD, CDB were shewn to be equal to two right angles; and one of them ABD is a right angle: therefore the remaining angle CDB is a right angle. Now since AB is equal to DE by const^n, BD common to two triangles ABD, BDE, and the right angle ABD equal to the right angle BDE, since all right angles are equal (Ax. 11); therefore these two triangles have the two sides AB, BD respectively equal to the two sides ED, DB and the included angle ABD equal to the included angle EDB. Therefore they are equal in every respect (i. 4); and hence the base AD is equal to the base BE. And AB is equal to DE, and AE common to the two triangles ABE, ADE: therefore these two triangles have the three sides AB, BE, EA respectively equal to the three sides ED, DA, AE. Therefore they are equal in every respect (i. 8); and hence the angle ABE is equal to the angle EDA. But ABE is a right angle; therefore also EDA is a right angle. And EDB is a right angle; hence ED is at right angles to each of the straight lines BD, AD at the point of their intersection D. Therefore ED is perpendicular to the plane through BD, DA (xi. 5); and CD, which has been shewn to be in the same plane with AB, BD, DA, is drawn meeting ED in the plane through BD, DA; therefore by the def^n of a straight line being perpendicular

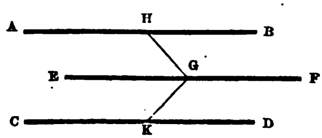
to a plane, the angle EDC is a right angle. But CDB is a right angle; hence CD is at right angles to each of the two straight lines DB, DE at the point of their intersection D: therefore CD is perpendicular (xi. 4) to the plane through DB, DE, that is, to the plane FG, to which AB is at right angles. Which was to be proved.

PROP. IX. THEOR.

Two straight lines, which are each of them parallel to the same straight line, and are not both in the same plane with it, shall be parallel to one another.

Let AB, CD be each of them parallel to the same straight line EF, and be not both in the same plane with it, so that AB, EF, CD are not all in one plane. Then AB shall be parallel to CD.

In EF take any point G; from G draw (i. 12) in the plane, in which the parallels AB, EF are, GH at right angles to EF cutting AB in H; and from G draw, in the plane, in which the parallels EF, CD are, GK at right angles to EF, cutting CD in K.



Because EF is at right angles by constⁿ to each of the two straight lines GH, GK at the point of their intersection G; therefore it is perpendicular to the plane passing through them (xi. 4), or the plane HGK. Now AB is parallel to EF by hypˢ: and EF has been just shewn to be perpendicular to the plane HGK: therefore also AB is perpendicular to this plane (xi. 8). In the same manner it may be shewn that CD is perpendicular to the plane HGK; hence AB, CD are each of them perpendicular to the plane HGK: therefore AB is parallel (xi. 6) to CD. Which was to be proved.

PROP. X. THEOR.

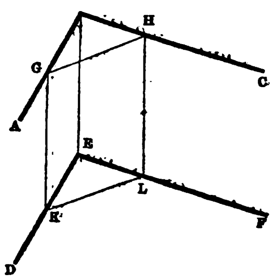
If two straight lines meeting one another be respectively parallel to two others meeting one another, but are not in the same plane with the first two: then

the first two and the other two shall include equal angles.

Let the two straight lines AB, BC, meeting one another in B, be respectively parallel to the two DE, EF, meeting one another in E; that is, let AB be parallel to DE, and BC parallel to EF; and let the plane through AB, BC be not the same with that through DE, EF. Then the angle ABC included by AB, BC shall be equal to the angle DEF included by DE, EF.

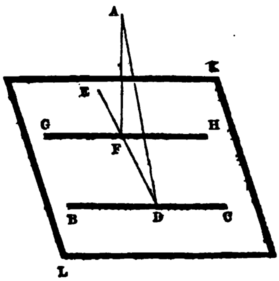
In BA take any point G, and from BC, ED, EF cut off (i. 3) BH, EK, EL equal to BG. Join GK, BE, HL, GH, KL.

By constr the four straight lines BG, BH, EK, EL are all equal. And BA is parallel to ED by hyps; hence BG is equal and parallel to EK; and they are joined towards the same parts, viz. towards G, K and towards B, E by the straight lines GK, BE: therefore GK is equal and parallel (i. 33) to BE. For like reason, HL is equal and parallel to BE; hence GK, HL are both equal to BE, and both parallel to BE. Now since things that are equal to the same thing are equal to one another (Ax. 1), therefore GK is equal to HL; and since straight lines that are parallel to the same straight line and not both in the same plane with it are parallel to one another (i. 9), therefore GK is parallel to HL. Hence GK, HL are equal and parallel straight lines; and they are joined towards the same parts, viz. towards G, H and towards K, L, by the straight lines GH, KL: therefore GH, KL are themselves equal and parallel. Also BG, EK, BH, EL are all equal; therefore the two triangles BGH, EKL have the three sides GB, BH, HG respectively equal to the three sides GE, EL, LK. Therefore they are equal in every respect (i. 8); and hence the angle GBH is equal to the angle KEL, that is, the angle included by AB, BC is equal to that included by DE, EF. Which was to be proved.



PROP. XI. PROB.

To draw a straight line perpendicular to a given plane from a given point without the plane.



Let KL be the given plane, and A the given point without it. It is required to draw from A a straight line perpendicular to the plane KL.

In the plane KL draw any straight line BC; in the plane which passes through BC and the point A, from A draw (i. 12) AD at right angles to BC; and from D draw (i. 11), in the plane KL, DE at right angles to BC. Then if AD is also at right angles to DE, AD is at right angles to each of the two straight lines BC, DE at the point of their intersection D, and is therefore perpendicular to the plane through ED, BC (xi. 4); that is, from A has been drawn a straight line AD perpendicular to the plane KL, and the thing required is already done. But if AD is not perpendicular to DE; in the plane passing through DE and the point A, from A draw AF perpendicular to DE. Then AF shall be perpendicular to the plane KL.

Through F in the plane KL draw (i. 31) GH parallel to BC.

By constn BC is at right angles to each of the two straight lines DA, DE at the point of their intersection D; therefore it is perpendicular to the plane through DA, DE (xi. 4). And GH is parallel to BC by constn; therefore also (xi. 8) GH is at right angles to the plane through DA, DE. But AF is in the plane through DA, DE, because each of the three straight lines AF, DE, DA cuts the other two, and they are therefore (xi. 2) all in one plane; and AF is drawn meeting GH in this plane: therefore by the defn of a straight line being perpendicular to a plane (xi. Def. 3), GFA is a right angle. Now, AFE is a right angle by constn; hence AF is at right angles to each of the two straight

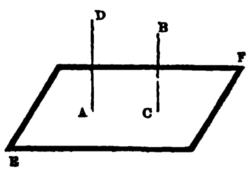
lines ED, GF at the point of their intersection F. Therefore AF is perpendicular to the plane through ED, GH, or the plane KL; that is, from the given point A without the given plane KL has been drawn AF perpendicular to that plane. Which was to be done.

PROP. XII. PROB.

To draw a straight line perpendicular to a given plane from a given point in the plane.

Let EF be the given plane, and A the given point in it. It is required to draw from A a straight line perpendicular to the plane.

Take any point B without the plane EF; and from B draw (xi. 11) BC perpendicular to the plane, cutting it in C. Then if C coincides with A, from A has been drawn a straight line CB or AB perpendicular to the plane; and what was required is already done. But if C does not coincide with A, in the plane through BC and the point A draw (i. 31) AD parallel to BC. Then AD shall be perpendicular to the plane EF.



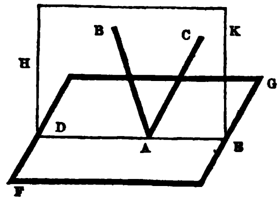
Because by constⁿ AD, CB are parallel straight lines, and one of them BC is perpendicular to the plane EF, therefore the other AD is also perpendicular (xi. 8) to it. Hence from the given point A has been drawn a straight line AD perpendicular to the given plane EF. Which was to be done.

PROP. XIII. THEOR.

From the same point in a given plane, there cannot be two straight lines perpendicular to the plane on the same side of it. And there can be but one perpendicular to a plane from the same point without the plane.

I. From the same point in a given plane, there cannot be two straight lines perpendicular to the plane on the same side of it.

For if there can: let, if possible, from the point A in the plane FG the two straight lines AB, AC be perpendicular to the plane on the same side of it.

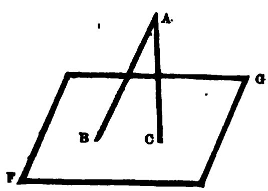


Through AB, AC draw the plane HK; and since the common section of this plane with the plane FG is a straight line (xi. 3) through A, let it be the straight line DAE.

By constr[n] the three straight lines AB, AC, DAE are in one plane HK. Now, because CA by hyp[s] is perpendicular to the plane FG, and EA is drawn meeting it in this plane; therefore by the def[n] of a straight line being perpendicular to a plane (xi. Def. 3), CAE is a right angle. For like reason, BAE is a right angle; and all right angles are equal (Ax. 11): therefore the angle CAE is equal to the angle BAE. But because BA, CA, AE are all in one plane HK, the angle CAE is a part of the angle BAE; hence the part CAE is equal to the whole BAE: which is impossible (Ax. 9). Therefore from the same point in a given plane there cannot be drawn two straight lines perpendicular to the plane on the same side of it. Which was to be proved.

II. There can be but one perpendicular to a plane from the same point without it.

For if there can be more than one; let, if possible, from the same point A without the plane FG the two straight lines AB, AC be perpendicular to the plane.



Then, since AB, AC are both perpendicular to the same plane FG, they are parallel (xi. 6) to one another. But they cannot be parallel (Def. 35), because they meet in the point A: which is impossible. Therefore there can be but one perpendicular

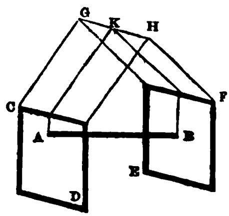
to a plane from the same point without it. Which was to be proved.

PROP. XIV. THEOR.

Two planes, to each of which the same straight line is perpendicular, shall be parallel to one another.

Let the same straight line AB be perpendicular to each of the two planes CD, EF, meeting CD at A and EF at B. Then the planes CD, EF shall be parallel.

For if they are not parallel, they will meet one another if produced far enough in some direction or other (xi. Def. 8): let them, if possible, be produced until they meet, and let their common section, which is a straight line (xi. 3), be GH. In GH take any point K, and join AK in the plane DCH and BK in the plane EFG.



Because by hyp[s] BA is perpendicular to the plane CD, and KA is drawn meeting it in that plane; therefore by the def[n] of a straight line being perpendicular to a plane (xi. Def. 3), the angle KAB is a right angle. Similarly it may be shewn that the angle KBA is a right angle. Hence the angles KAB, KBA are equal to two right angles. But because each of the three straight lines KA, AB, BK cuts the other two, they are all in one plane (xi. 2), and form a triangle KAB; and two of its angles KAB, KBA have been just shewn to be equal to two right angles: which is impossible (i. 17). Therefore the planes CD, EF, if produced ever so far every way, shall never meet; that is, by the def[n] of parallel planes, the planes CD, EF are parallel. Which was to be proved.

PROP. XV. THEOR.

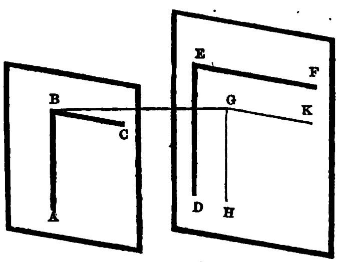
If two straight lines meeting one another be respectively parallel to two others, which meet one another, but are not in the same plane with the first two: then

the plane passing through the first two shall be parallel to the plane passing through the other two.

Let the two straight lines AB, BC meeting one another in B be respectively parallel to the two DE, DF meeting one another in E; that is, let AB be parallel to DE, and BC to DF; and let the plane through AB, BC be not the same with that through DE, EF. Then the planes through AB, BC and through DE, EF shall be parallel.

From B draw (XI. 11) BG perpendicular to the plane through DE, EF; and from G, the point where BG meets the plane, draw (I. 31) in it GH parallel to ED, and GK parallel to EF.

Because BG by constⁿ is at right angles to the plane through DE, EF; and HG, KG are drawn meeting it in this plane: therefore by the defⁿ of a straight line being perpendicular to a plane (XI. Def. 3), each of the angles BGH, BGK is a right angle. Now, because BA is parallel to ED by hyp^s, GH parallel to ED by constⁿ; and straight lines which are parallel to the same straight line and not both in the same plane with it are parallel to one another (XI. 9): therefore BA is parallel to GH. And

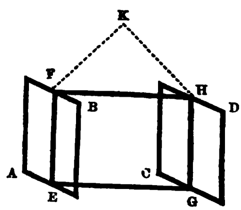


because BG cuts the parallels BA, GH in G, H, the two interior angles on the same side of it GBA, BGH are equal (I. 29) to two right angles; but one of them BGH is a right angle: therefore the other GBA is a right angle. In the same way it may be shewn that GBC is a right angle; hence GB is at right angles to each of the two straight lines AB, CB at the point of their intersection B. Therefore GB is at right angles to the plane through AB, BC (XI. 4); and by constⁿ it is at right angles to the plane through DE, EF. But planes to each of which the same straight line is perpendicular are parallel (XI. 14): there-

fore the plane through AB, BC is parallel to the plane through DE, EF. Which was to be proved.

PROP. XVI. THEOR.

If two parallel planes be each of them cut by a third plane: then their common sections with it shall be parallel.



Let the two parallel planes AB, CD be each of them cut by the plane EFHG; and let their common sections with it, which are straight lines (XI. 3), be EF, GH. Then EF shall be parallel to GH.

For if not: let, if possible, EF, GH be not parallel. Then they being produced far enough must meet either towards F, H, or towards E, G. First let them be produced and meet towards F, H in the point K.

Then since EF is in the plane AB and has been produced to K, every point in the straight line EFK is in the plane AB, or in the plane AB produced; but K is a point in EFK: therefore K is a point in the plane AB produced. In like manner it may be shewn that K is a point in the plane CD produced; hence the planes AB, CD when produced have a common point K, and therefore meet one another; but they never meet one another, if produced ever so far every way (XI. Def. 8), because they are parallel by hyp^s: which is impossible. Therefore EF and GH, being produced, do not meet towards F, H. In like manner it may be proved that they, being produced, do not meet towards E, G. Hence EF, GH in the plane EFHG being produced ever so far both ways do not meet: therefore by the def^n of parallel straight lines (Def. 35), EF is parallel to GH. Which was to be proved.

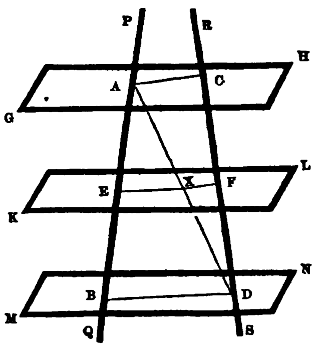
PROP. XVII. THEOR.

If any two straight lines, whether in one or in different planes, be each of them cut by three parallel

planes; then the four parts cut off from them at the points of section shall be proportional.

Let PQ, RS be any two straight lines whether in one or in different planes; and let the three parallel planes GH, KL, MN cut PQ in the points A, E, B, and RS in the points C, F, D so that from PQ are cut off the two parts AE, EB, and from RS the two parts CF, FD. Then AE shall be to EB as CF is to FD.

Join AC, BD, AD; and let X be the point where AD cuts the plane KL. Join EX, XF.



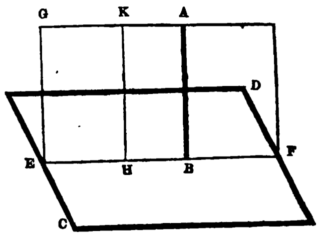
Then because BA, DA cut one another in A, they are in one plane (XI. 2); and E, X and B, D are points in this plane, therefore by the defn of a plane (Def. 7) EX, BD the straight lines joining them are both in the plane through AB, AD. And EX, BD are drawn in the planes KL, MN: hence EX, BD are the common sections of the plane through BA, AD with the two parallel planes KL, MN. But if two parallel planes be cut by another plane, the common sections with it are parallels (XI. 16): therefore EX is parallel to BD. In the same way it may be shewn that XF, AC are both in one plane, that through DA, DC, and parallel to one another. Now, because EX is drawn parallel to one side BD of the triangle ABD, cutting the other sides AB, AD in E and X, therefore AE is to EB as AX to XD (VI. 2). Again, because XF is drawn parallel to one side AC of the triangle ACD, cutting the other sides AD, CD in X and F, therefore CF is to FD as AX to XD. But it was shewn that AE is to EB as AX to XD; and ratios that are the same to the same ratio are the same to one another (V. 11): therefore AE is to EB as CF to FD. Which was to be proved.

PROP. XVIII. THEOR.

If a straight line be at right angles to a plane: then every plane which passes through it shall be at right angles to that plane.

Let the straight line AB be at right angles to the plane CD. Then every plane, which passes through AB, shall be at right angles to the plane CD.

Take any plane GF passing through AB, and let its common section with the plane CD, which is a straight line (xi. 3), be EBF. Take any point H in EF, not coinciding with B, and from H draw (i. 11), in the plane FG, HK at right angles to EF.



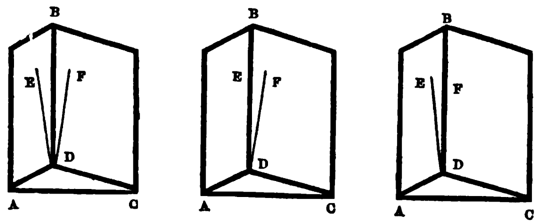
Because AB by hyp[s] is perpendicular to the plane CD, and EB is drawn meeting it in this plane; therefore by the def[n] of a straight line being perpendicular to a plane (xi. Def. 3), EBA is a right angle. But KHB is a right angle by const[n]; hence HB cutting HK, AB, in the plane GF, in H and B makes the two interior angles KHD, ABH on the same side of HB equal to two right angles. Therefore HK is parallel (i. 28) to AB; but of these two parallel straight lines HK, AB one of them AB is at right angles to the plane CD: therefore the other HK also is at right angles (xi. 8) to the plane CD. But HK is any straight line drawn in the plane FG perpendicular to EF: hence the straight lines drawn in one of the planes FG, CD perpendicular to their common section EF are perpendicular to the other plane. Therefore by the def[n] of a plane being perpendicular to a plane (xi. Def. 4), the plane FG is perpendicular to the plane CD. In like manner it may be shewn that every other plane which passes through AB is perpendicular to the plane CD. Which was to be proved.

PROP. XIX. THEOR.

If two planes cutting one another be each of them perpendicular to a third plane: then their common section shall be perpendicular to the same plane.

Let the two planes AB, BC, which cut one another, and have for their common section the straight line (xi. 3) BD, be each of them perpendicular to a third plane, with which their common sections are the straight lines AD, DC respectively. Then shall BD be perpendicular to the third plane ADC.

For if not: let, if possible, BD be not perpendicular to the plane ADC at D. From D draw (i. 11) in the plane AB, DE at right angles to AD; and from D draw, in the plane BC, DF at right angles to DC. Of the straight lines DE, DF though one may, yet both cannot coincide with DB; for if they did, then DB would be perpendicular to each of the straight lines DA, DC at the point of their intersection D, and therefore (xi. 4) to the plane ADC, which it is supposed not to be.



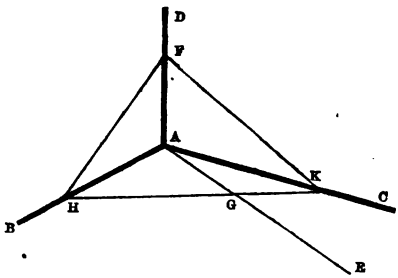
Because the plane AB is perpendicular to the plane ADC by hyp^s, and DE is drawn in the plane AB perpendicular to their common section AD; therefore by the def^n of a plane being perpendicular to a plane (xi. Def. 4), DE is perpendicular to the plane ADC. In like manner it may be shewn that DF is perpendicular to the plane ADC. But DE, DF are in different planes AB, BC, and can only coincide by both coinciding with DB, which it has been shewn they cannot do; hence from the same point D in the plane ADC there are drawn two different straight lines

DE, DF perpendicular to the plane ADC on the same side of it: which is impossible (XI. 13). Therefore DB cannot be not perpendicular to the plane ADC; that is, BD is perpendicular to the third plane ADC. Which was to be proved.

PROP. XX. THEOR.

If a solid angle be constituted by three plane angles: then any two of them shall be together greater than the third.

Let the solid angle at A be constituted by the three plane angles BAC, CAD, DAB. Then any two of them shall be together greater than the third.



The three angles BAC, CAD, DAB are either all equal, or not all equal. If they are all equal, it is manifest that any two of them are together greater than the third. But if they are not all equal; there are either two equal angles which are each greater than the third, or else one angle is greater than each of the other two: so that in every case there is an angle not less than either of the other two, and greater than one of them. Let BAC be that angle, which is not less than either of BAD, DAC and greater than one of them, BAD. At the point A in the straight line AB make (I. 23) in the plane of the angle BAC, the angle BAE equal to the angle BAD, AE falling between AB and AC, since BAD is supposed less than BAC. In AD take any point F; from AE cut off (I. 3) AG equal to AF; through G draw in the plane BAC any straight line HGK, cutting AB in H and AC in K; and join FH, FK.

By const^{n} the angle HAF is equal to the angle HAG and AF to AG, and AH is common to the two triangles FAH, GAH; therefore these two triangles have the two sides FA, AH respectively equal to the two sides GA, AH, and the included angle FAH equal to the included angle GAH. Therefore they are equal in every respect (i. 4); and hence the base FH is equal to the base GH. Now since each of the three straight lines FH, HK, KF cuts the other two, they are in one plane (xi. 2) and form the triangle FHK, any two of whose sides are greater than the third (i. 20); therefore HF, FK are greater than HK. And it has been shewn that HF is equal to HG; hence, taking away equals from unequals, the remainder FK is greater (Ax. 5) than the remainder GK. Also FA is equal to AG, and AK common to the two triangles FAK, GAK; hence these two triangles have the two sides FA, AK respectively equal to the two sides GA, AK, but the base FK greater than the base GK. Therefore the angle FAK included by the sides of that which has the greater base FK is greater (i. 25) than the angle GAK included by the sides of the other. But the angle FAH is equal to the angle GAH; hence adding equals to unequals, the two angles FAH, FAK are greater (Ax. 4) than the two angles HAG, GAK, that is, than the angle HAK. Thus it has been shewn that the two angles BAD, CAD are greater than the angle BAC; and since BAC is supposed not less than either of the two BAD, CAD, therefore BAC together with either of them is greater than the other. Hence any two of the three angles BAC, BAD, CAD are together greater than the third. Which was to be proved.

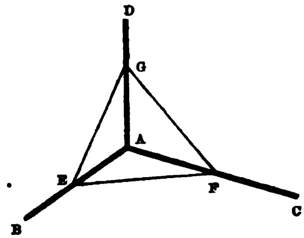
PROP. XXI. THEOR.

The plane angles which constitute any solid angle shall be together less than four right angles.

There are two cases according as the number of the plane angles is three or more than three.

I. Let the solid angle at A be constituted by three plane angles BAC, CAD, DAB. Then these three together shall be less than four right angles.

In the straight lines AB, AC, AD take any points E, F, G: and join EF, FG, GE.

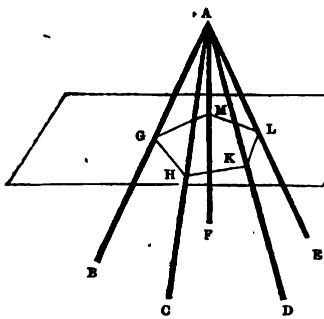


Then because each of the three straight lines EF, FG, GE cuts the other two, they are all in one plane (xi. 2) and form a triangle EFG. Now since the solid angle at E is contained by three plane angles FEA, AEG, FEG, the two FEA, AEG are greater than the third FEG. For like reason, the angles EFA, AFG are greater than the angle EFG; and the angles FGA, EGA greater than the angle FGE. Hence, adding these unequals together, the six angles FEA, AEG, AGE, AGF, GFA, AFE are greater than the three angles FEG, EFG, FGE; but FEG, EFG, FGE, the three angles of the triangle FEG, are equal to two right angles (i. 32): therefore the six angles FEA, AEG, AGE, AGF, GFA, AFE are greater than two right angles. Again, because the three angles of each of the triangles AFE, AFG, EGA are equal to two right angles; therefore the nine angles of these three triangles, viz. FEA, EAF, AFE, AFG, FAG, FGA, AGE, GEA, EAG are equal to six right angles. But of these nine angles, six, viz. FEA, AEG, AGE, AGF, GFA, AFE have been shewn to be greater than two right angles: therefore the remaining three, viz. EAG, GAF, FAE are less than four right angles. Hence the three angles BAC, CAD, DAB which constitute the solid angle at A, are less than four right angles. Which was to be proved.

II. Let the solid angle at A be constituted by any number of plane angles, BAC, CAD, DAE, EAF, FAB. Then these together shall be less than four right angles.

Take any plane cutting each of the planes in which the plane angles are; and let its common sections with them be GH, HK, KL, LM, MG.

Then GH, HK, KL, LM, MG are all straight lines (xi. 3); and as they meet one another and are in one plane, they



form a polygon GHKLM, which has as many sides as there are plane angles constituting the solid angle at A. Also each of the sides of the polygon as GH is the base of a triangle in the plane of one of the plane angles, and having its opposite angular point at A. Now since the solid angle at G is constituted by the three plane angles AGM, AGH, MGH, the two AGM, AGH are greater (XI. 20) than the third MGH. For the same reason the two plane angles at each of the points H, K, L, M, which are at the bases of the triangles having the common angular point at A, are each greater than the third angle at the same point, which is one of the angles of the polygon GHKLM. Therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon GHKLM. To each of these unequals add four right angles; therefore all the angles at the bases of the triangles together with four right angles are greater (Ax. 4) than all the angles of the polygon together with four right angles. But all the angles of the polygon GHKLM together with four right angles are equal to twice as many right angles as the polygon has sides (I. 32. Cor. 1), that is, as there are plane angles constituting the solid angle at A; therefore likewise all the angles at the bases of the triangles together with four right angles are greater than twice as many right angles as there are plane angles constituting the solid angle at A. Now because the three angles of each of the triangles are equal to two right angles (I. 32), all the angles of the triangles are

equal to twice as many right angles as there are triangles, that is, as there are plane angles constituting the solid angle at A: hence all the angles at the bases of the triangles together with four right angles are greater than all the angles of the triangles, that is, than all the angles at the bases of the triangles together with all the angles at A. From each of these unequals take away the common angles at the bases of the triangles; then the remaining four right angles are greater (Ax. 5) than the angles of the triangles at A. That is, the plane angles BAC, CAD, DAE, EAF, FAB, which constitute the solid angle at A, are together less than four right angles. Which was to be proved.

OBS. The proof (I.) is but a particular case of the general demonstration (II.), and may be included in it.

THE

ELEMENTS OF EUCLID.

BOOK XII.

PROP. A.

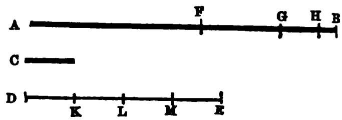
If from the greater of two unequal magnitudes there be taken away a part greater than its half, and from the remainder a part greater than its half; and so on: then this operation can be repeated until there is left a remainder less than the less of the two unequal magnitudes.

OBS. This propⁿ is required for the proof of Bk. xii. Prop. 2. It is the first propⁿ in the Tenth Book of the Elements.

Let AB and C be two unequal magnitudes of which AB is the greater; and from AB let there be taken away a part greater than its half, from the remainder a part greater than its half, and so on. Then this operation can be repeated until there is left a magnitude less than C.

Since the less magnitude C may be multiplied a sufficient number of times for its multiple to exceed the greater AB, let it be so multiplied, and let its multiple be DE, which is greater than AB. From AB take AH greater than its half; from the remainder FB take FG greater than its half; and repeat the operation till the number of divisions in AB is equal to the number of times C was multiplied. Let these divisions be AF, FG, GH, HB; and let DE be divided into magnitudes each equal to C, viz.

DK, KL, LM, ME, the number of these divisions being equal to that of the divisions of AB.



Because DE by constⁿ is greater than AB; and from DE is taken a part DK not greater than its half, while from AB is taken a part AF greater than its half: therefore the remainder KE is greater than the remainder FB. Again, because KE is greater than FB; and from KE is taken KL not greater than its half, and from FB is taken FG greater than its half; therefore the remainder LE is greater than the remainder GB. And proceeding in like manner for the corresponding divisions of DE and AE, we should have at last the remainder ME greater than the remainder HB. But ME is equal to C by constⁿ; therefore C is greater than HB. Hence the operation has been repeated until there remains a magnitude HB less than C. Which was to be proved.

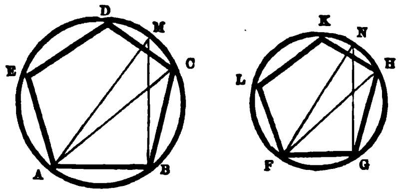
COR.—If from the greater of two unequal magnitudes there be taken away its half, and from the remainder its half; and so on: then this operation can be repeated until there is left a remainder less than the less of the two unequal magnitudes.

The proof of this is exactly similar to that of the propⁿ.

PROP. I. THEOR.

If similar polygons be inscribed in circles: then they shall be to one another as the squares of the diameters of the circles in which they are inscribed.

Let ABC, FGH be two circles; and in ABC, FGH let there be inscribed the similar polygons ABCDE, FGHKL. Then the polygon ABCDE shall be to the polygon FGHKL as the square of the diameter of the circle ABC is to the square of that of the circle FGH.



The angles at A, B, C, D, E being supposed those which by the defn (vi. Def. 1) of similar polygons are respectively equal to those at F, G, H, K, L, and the sides AB, BC, CD, DE, EA those which are homologous to FG, GH, HK, KL, LF, find (iii. 1) the centres of the circle ABC, FGH; and through A, F draw their diameters AM, FN. Join MB, NG, AC, FH.

Since the angle ABC is equal to the angle FGH, and AB is to BC as FG to GH (vi. Def. 1); therefore the two triangles ABC, FGH have the angle ABC equal to the angle FGH, and the sides about this pair of equal angles proportional. Therefore these two triangles are similar (vi. 6); and hence the angle ACB is equal to the angle FHG. But the angle AMB is equal to the angle ACB, because they are in the same segment AMCB; and the angle FNG is equal to the angle FHG for like reason: therefore the angle AMB is equal to the angle FNG. Also the angle ABM in the semicircle ABM, and the angle FGN in the semicircle FGN are right angles; and all right angles are equal: therefore the angle ABM is equal to the angle FGN. Hence the two triangles ABM, FGN have the two angles AMB, MBA respectively equal to the two angles FNG, NGF, and consequently (i. 32. Cor. A) the third angle BAM equal to the third angle GFN; therefore these two triangles are equiangular to one another. Therefore they are similar (vi. 4); and hence AB is to AM as FG to FN. Therefore, alternando (v. 16), AB is to FG as AM to FN; and the duplicate ratio of the ratio of AB to FG is the same as the duplicate ratio of the ratio of AM to FN. Now the polygons ABCDE, FGHKL are similar by hyps, and the squares described on AM, FN are similar, since regular polygons of the same number of sides are similar (vi. Def. 1. Obs. 3); and similar polygons have

to one another the duplicate ratio of the ratio of their homologous sides (vi. 20): therefore ABCDE has to FGHKL the duplicate ratio of the ratio that AB has to FG, and the square described on AM has to the square described on FN the duplicate ratio of the ratio that AM has to FN. But it was shewn that the duplicate ratio of the ratio of AB to FG is the same as the duplicate ratio of the ratio of AM to FN; and ratios that are the same ratio are the same to one another (v. 11): therefore ABCDE has to FGHKL the same ratio which the square described on AM has to the square described on FN; that is, the polygon ABCDE is to the polygon FGHKL as the square of the diameter of the circle ABC to the square of the diameter of the circle FGH. Which was to be proved.

COR.—Similar polygons shall be to one another as the squares of either pair of their homologous sides.

For if ABCDE, FGHKL be supposed any similar polygons whatever, not necessarily inscribed in circles, and AB, FG a pair of homologous sides; it may be shewn in like manner as in the propn that ABCDE has to FGHKL the duplicate ratio of the ratio of AB to FG, and the square of AB has to the square of FG the duplicate ratio of the ratio of AB to FG, and therefore that the polygon ABCDE is to the polygon FGHKL as the square of AB is to the square of FG. Which was to be proved.

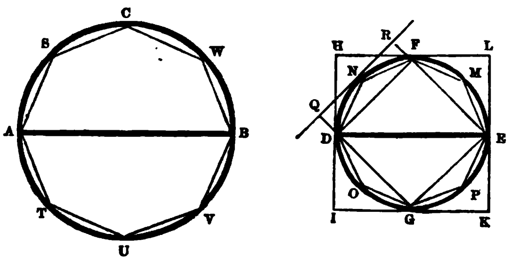
PROP. II. THEOR.

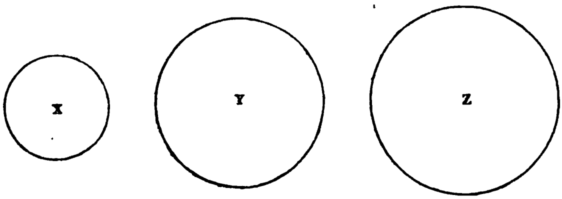
Circles shall be to one another as the squares of their diameters.

Let ABC, DEF be two circles; and AB, DE diameters of them respectively. Then the square of AB shall be to the square of DE as the circle ABC is to the circle DEF.

The square of AB must be to the square of DE as the circle ABC is to some space or other. If this space be not equal to the circle DEF, it must be either less than it, or greater than it.

I. Let, if possible, the square of AB be to the square of DE as the circle ABC is to a space X, less than the circle DEF.





In the circle DEF inscribe (iv. 6) the square EFDG, so as to have DE for one of its diagonals, and through E, F, D, G draw (iii. 17) straight lines touching the circle: these will form (iv. 7) the square HIKL circumscribed about the circle DEF, and LD, DK will be each rectangles. Then since the triangle FDE and the parallelogram LD are on the same base DE and between the same parallels HL, DE, LD is double (i. 41) of FDE. In like manner DK is double of DGE; therefore the whole HIKL is double of the whole EFDG. But HIKL is greater than the circle DEF; therefore the double of EFDG is greater than the circle DEF, and therefore EFDG is greater than half the circle DEF:

Again, bisect (iii. 30) the arcs FD, DG, GE, EF in M, N, O, P, and join ND, DO, OG, GP, PE, EM, MF, FN. Through N draw a straight line touching the circle; from D draw (i. 11) DQ perpendicular to it, and through F draw (i. 31) FR parallel to DQ cutting the touching line in R. Then RQ is at right angles to the straight line drawn from N

through the centre (iii. 18); and from the problem (iii. 30) of bisecting an arc, this straight line bisects FD at right angles: therefore QR, DF, which are cut by this line, make the two interior angles on the same side together equal to two right angles, and are therefore parallel (i. 28). Hence QDFR is a parallelogram (Def. A); and it may be shewn as before, that the triangle DNF is greater than half the segment DNF. In like manner, each of the other triangles DOG, GPE, EMF is double of the segment in which it is; and therefore all the triangles DNF, DOG, GPE, EMF together are greater than half of the segments DNF, DOG, GPE, EMF together. And a like result would appear each time that the arcs are bisected, and their extremities joined with the points of bisection:

By hyp[s] the space X is less than the circle DEF; and therefore the circle DEF and the excess of the circle DEF above the space X are two unequal magnitudes, of which the circle DEF is the greater. Now if from the greater of these two unequal magnitudes, viz. the circle DEF, there be taken a part greater than its half, and from the remainder a part greater than its half, and so on; then this operation can be repeated (xii. A) until there is left a remainder which is less than the less of the two unequal magnitudes, viz. the excess of the circle DEF above the space X. Let such operations be performed: that is, from the circle DEF take away the inscribed square EGBDF, which has been shewn to be greater than its half; from the remainder, viz. the segments FND, DOG, GPE, EMF together take away the triangles FND, DOG, GPE, EMF, which have been shewn to be together greater than its half; and so on; and let the remainder after the last operation, which is less than the excess of the circle DEF above the space X, be the segments FN, ND, DO, OG, GP, PE, EM, MF. But all these segments together are the excess of the circle DEF above the inscribed regular polygon FNDOGPEM; therefore the excess of the circle DEF above the polygon FNDOGPEM is less than the excess of the circle DEF above the space X. Therefore the polygon FNDOGPEM is greater than the space X:

Next, in the circle ABC let there be inscribed a regular polygon CSATUVBW of the same number of sides as the

polygon FNDOGPEM, by inscribing a square, and successively bisecting the arcs of the circle, and joining the points of bisection with their extremities: then these two regular polygons will be similar (vi. Def. 1. Obs. 3). But similar polygons inscribed in circles are as the squares of their diameters (xii. 1): therefore the square of AB is to the square of DE as the polygon CSATUVBW is to the polygon FNDOGPEM. Now, by hyp^s the square of AB is to the square of DE as the circle ABC is to the space X; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore the circle ABC is to the space X as the polygon CSATUVBW is to the polygon FNDOGPEM. But the circle ABC is greater (Ax. 9) than the polygon CSATUVBW; therefore the space X is greater (v. 14) than the polygon FNDOGPEM, that is, the polygon FNDOGPEM is less than the space X. But it has been proved to be greater than the space X: which is impossible:

Therefore the square of AB is not to the square of DE as the circle ABC to any space less than the circle DEF; and generally the square of the diameter of one circle is not to the square of the diameter of another circle as the first circle is to any space less than the second circle.

II. Let, if possible, the square of AB be to the square of DE as the circle ABC is to a space Y greater than the circle DEF.

Then, invertendo (v. B), the square of DE is to the square of AB as the space Y is to the circle ABC. But the space Y is to the circle ABC as the circle DEF is to some space or other, as Z; and the space Y is by hyp^s greater than the circle DEF: therefore also the circle ABC is greater (v. 14) than the space Z. Now since the square of DE is to the square of AB as the space Y is to the circle ABC, and the space Y is to the circle ABC as the circle DEF to a space Z less than the circle ABC; and ratios that are the same to the same ratio are the same to one another (v. 11): therefore the square of DE is to the square of AB as the circle DEF to a space Z less than the circle DEF; which has been shewn in Part I. of the proof to be impossible:

Therefore the square of AB is not to the square of DE

as the circle ABC is to any space greater than the circle DEF.

Hence since the square of AB must be to the square of DE as the circle NBC is to some space or other; and since it has been shewn that this space is neither less nor greater than the circle DEF: therefore it is equal to the circle DEF; that is, the square of AB is to the square of DE as the circle ABC is to the circle DEF. Which was to be proved.

THE

ELEMENTS OF EUCLID.

APPENDIX A.

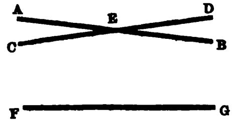
PROFESSOR PLAYFAIR, in his "Elements of Geometry" (Glasgow, 1795), substituted the axiom that "two straight lines, which cut one another, cannot be each of them parallel to the same straight line," for Bk. I. Ax. 12, as given in the text. If this method be adopted, the above may be made Axiom A. of the 1st Book, and the assertion of Euclid's 12th Axiom proved by means of it; this will form Bk. I. Prop. A., and may be placed conveniently after Prop. XXVIII. The prop^ns then in which Ax. 12 is introduced may either be proved as in the text, reference being made to Prop. A. instead of to Ax. 12; or their proofs may be so modified as to depend immediately on Playfair's Axiom. We shall enunciate Ax. A., state and prove Prop. A., and prove Bk. I. Prop. XXIX. immediately by help of Ax. A., according to Playfair.

AXIOM A.

Two straight lines, which cut one another, cannot be each of them parallel to the same straight line.

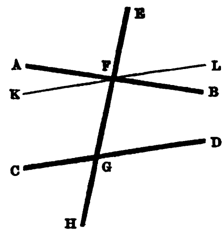
OBS. To illustrate this; let AB, CD be two straight lines which cut one another in the point E. Then Ax. A asserts that AB, CD shall not be each of them parallel to the same straight line FG.

Either one of the two AB, CD may be parallel to FG; but both cannot.



BK. I. PROP. A.

If a straight line cut two straight lines, so as to make the two interior angles on the same side of it together less than two right angles: then these two straight lines, being continually produced, shall at length meet on that side of the cutting line on which are the angles that are together less than two right angles.



Let the straight line EFGH cut AB, CD in F, G, so as to make the two interior angles BFG, FGD on the same side of EH (viz. that side towards B, D,) together less than two right angles. Then AB, CD, being continually produced, shall at length meet in some point on the side of EH towards B, D.

For if they do not meet on the side of EH towards B, D; they must either meet if produced on the side of EH towards A, C, or else be parallel.

I. Let, if possible, AB, CD meet if produced on the side of EH towards A, C.

Then the angles GFA, FGC are two angles of a triangle, and therefore are together less (I. 17) than two right angles. But since GF makes with AB the adjacent angles AFG, BFG, these two angles are equal to two right angles; and for like reason the two angles CGF, FGD are equal to two right angles: hence, adding equals to equals, the four angles AFG, FGC, FGD, GFB are equal to four right angles. But two of them, AFG, FGC have been shewn to be less than two right angles, therefore the other two BFG, FGD are greater than two right angles; and by hyp[s] they are also less: which is impossible. Therefore AB, CD do not, if produced, meet on the side of EH towards A, C.

II. Let, if possible, AB, CD be parallel.

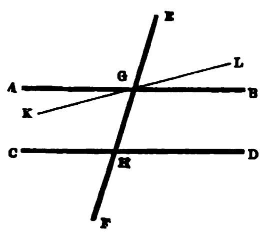
At the point F in the straight line GF make the angle GFL equal to the angle FGC; and produce LF to K.

Then, because FG makes with CD the adjacent angles FGC, FGD, these two angles are equal to two right angles; and the angle FGC is equal to the angle GFL by constn: therefore the two angles GFL, FGD are equal to two right angles. But the two angles GFB, FGD are by hyps less than two right angles; therefore the two angles GFL, FGD are greater than the two angles BFG, FGD. From each of these unequals take away the common angle FGD; therefore the remaining angle GFL is greater than the remaining angle GFB, and FL does not coincide with FB; that is, AB, KL are two straight lines, cutting one another in F. But since HE, cutting the two straight lines KL, CD in G, F makes the alternate angles CGF, GFL equal, KL is parallel (i. 27) to CD; and AB is supposed parallel to CD. That is, the two straight lines AB, CD, which cut one another in F, are each of them parallel to the same straight line CD: which by the axiom (Ax. A) is impossible. Therefore AB, CD are not parallel.

Hence, since it has been shewn, that AB, CD neither, if produced, meet on the side of EH towards A, C, nor are parallel, AB, CD must, if continually produced, meet on the side of EH towards B, D. Which was to be proved.

The following is the proof of Bk. I. Prop. XXIX. as given by Playfair, and depending immediately on Axiom A.

BK. I. PROP. XXIX.



Enunciate as in the text, and proceed thus:—

For if AGH, GHD be not equal; let them, if possible, be unequal. At the point G in the straight line HG make the angle HGK equal to the angle GHD; and produce KG to L.

Then by constn the angle HGK is unequal to the angle HGA; therefore GK and GA do not coincide, and AB, KL are two straight lines cutting one another in G. Now

because EF cutting KL, CD in G, H makes the alternate angles KGH, GHD equal, therefore KL is parallel (i. 28) to CD; and AB is by hyp^s parallel to CD. That is, the two straight lines AB, KL, which cut one another in G are each of them parallel to the same straight line CD: which by the axiom (Ax. A) is impossible. Therefore the angles AGH, GHD are not unequal, that is, the alternate angles AGH, GHD are equal. In like manner it may be shewn that the alternate angles BGH, GHC are equal. Which was to be proved.

The proofs of Parts II. and III. are the same as those in the text.

Playfair deduces what is given above as Prop. A from this prop^n, to which he makes it a Corollary.

APPENDIX B.

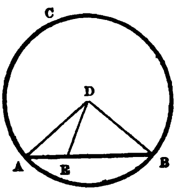
If we lay down the following as an Axiom:—

If the distance of a point from the centre of a circle be less than the radius of the circle, the point is within the circle; and if the distance of a point from the centre of a circle is greater than the radius, the point is without the circle:

The 2nd and 18th Prop^{ns} of Bk. III. may be proved by a direct proof, instead of the ex absurdo demonstrations given in the text, as we shall now shew; and the above axiom will be referred to as Bk. III. Ax. A. These methods are due to Commandine, who published the fifteen books of Euclid's Elements in Latin, at Pisauri, fol., in the year 1572.

BK. III. PROP. II.

Let ABC be a circle and A, B any two points in the circumference. Then the straight line AB, which joins them, shall fall within the circle.



In AB take any point E; find D the centre of the circle (iii. 1); and join DA, DE, DB.

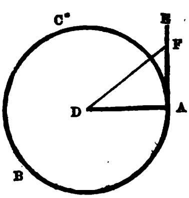
Because DA is equal to DB by the def^{n} of a circle, the angle DAB is equal (i. 5) to the angle DBA; and because the side AE of the triangle DAE is produced to B, the exterior angle DEB is greater than the interior and opposite angle DAB (i. 16): therefore the angle DEB is likewise greater than the angle DBE. But the greater angle of a triangle is subtended by the greater side (i. 19); therefore DB is greater than DE. That is, DE the distance of the point E from the centre D is less than the radius of the circle ABC; therefore by the axiom (iii. Ax. A) the point E is within the circle ABC. In like manner it may be shewn that every other point in AB is within the circle ABC, excepting A and B which are points in the circumference: therefore the straight line AB, which joins A and B, falls within the circle. Which was to be proved.

BK. III. PROP. XVI. THEOR.

Let ABC be a circle, of which D is the centre and DA any radius; and let the straight line AE be drawn at right angles to DA from its extremity A. Then:—

I. AE shall fall without the circle ABC.

In AE take any point F, and join DF.

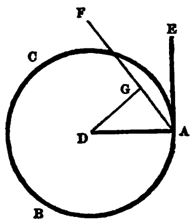


Because the angle DAF is a right angle by const^{n}, the angle DFA must be less than a right angle; for if it were either equal to or greater than a right angle, the two angles DAF, DFA of the triangle DAF would be not less than two right angles: which is impossible (i. 17). Hence the angle DAF is greater than the angle DFA; and the greater angle is subtended by the greater side (i. 19): therefore DF is greater than DA. That is, DF the distance of the point F from the centre D is greater than the radius of the circle ABC: therefor

by the axiom (iii. Ax. A), the point E is without the circle ABC. In like manner it may be shewn that every other point in AE is without the circle ABC, excepting the point A which by hyp^{s} is in the circumference: therefore the straight line AE falls without the circle. Which was to be proved.

II. No straight line can be drawn from A between AE and the radius AD, which shall not cut the circle ABC.

From A draw any straight line AF between AE and AD; and from D draw (i. 12) DG perpendicular to AF.



Because the angle DGA is a right angle by const^{n}, and the angle DAG is less (Ax. 9) than a right angle, since the angle DAF is a part of the right angle DAE: therefore the angle DGA is greater than the angle DAG. But the greater angle is subtended by the greater side (i. 19); therefore DA is greater than DG. That is, DG the distance of the point G from the centre D is less than the radius of the circle ABC: therefore by the axiom (iii. Ax. A) the point G is within the circle. Hence since the point A of the straight line AF is in the circumference, and the point G is a point within the circle, the straight line AF must cut the circle. In like manner it may be shewn that every other straight line drawn from A between AE and AD cuts the circle; that is, no straight line can be drawn from A between AE and the radius AD which shall not cut the circle ABC. Which was to be proved.

APPENDIX C.

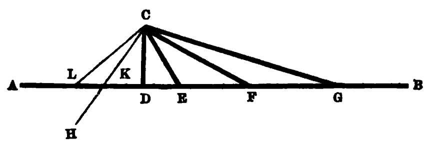
There is an analogous prop^{n} to Bk. III. Prop^{ns} VII. VIII. which is useful, inasmuch as it explains why in Bk. III. Def. 4 and Prop^{ns}. XIV. XV. the perpendicular from a point on a straight line is called the distance of the point from the line. Its enunciation and proof are as follows:—

BK. III. PROP. A. THEOR.

If any point be taken without a straight line: then

(1) of all the straight lines which can be drawn from it to the straight line, the greatest shall be that which is perpendicular to the straight line; and of the rest that which is nearer to the perpendicular shall be always greater than one more remote;

(2) from this point there can be drawn one and only one straight line to the straight line, equal to a given straight line drawn from it to the straight line, which shall be on the opposite side of the perpendicular.



Let AB be any straight line; C any point without it; and CD perpendicular to AB. Then:—

I. Of all the straight lines CD, CE, CF, CG that can be drawn from C to AB, CD shall be the least; and of the rest CE which is nearer to CD than CF shall be less than CF, and CF which is nearer to CD than CG shall be less than CG.

Because CDE is a right angle by hyp^{s}, the angle CED is less than a right angle; for if not, the two angles CDE, CED of the triangle CDE would be not less than two right angles (i. 17): which is impossible. Hence the angle CDE is greater than the angle CED; and the greater angle is subtended by the greater side (i. 19): therefore CE is greater than CD. Also, since CED is less than a right angle, the angle CEF is greater than a right angle; and it may be shewn as before that CFD is less than a right angle: therefore the angle CEF is greater than the angle CFE, and CF greater than CE. In like manner CG is

greater than CF. Hence of CD, CE, CF, CG the least is CD, CE is greater than CD, CF than CE, and CG than CF. Which was to be proved.

II. From C there can be drawn one and only straight line to AB equal to a given straight line CE drawn from C to AB, which will be on the opposite side of the perpendicular CD.

At the point C in the straight line DC make the angle DCH equal to the angle DCE; and let CH cut AD in K. Then CK shall be equal to CE.

Because the right angle CDK is equal to the right angle CDE since all right angles are equal (Ax. 11), the angle DCK to the angle DCE by const^n, and CD common to the two triangles DCK, DCE; therefore these two triangles have the two angles KDC, DCK respectively equal to the two angles EDC, ECD, and the side CD adjacent to the equal angles in each common. Therefore they are equal in every respect (i. 26); and hence CK is equal to CE. Also besides CK no other straight line can be drawn from C to AB equal to CE: for if there can, let it be CL. Then because CL is equal to CE, and CK to CE, and things that are equal to the same thing are equal to one another (Ax. 1); therefore CL is equal to CK, that is, the straight line CL drawn from C to AB, further from CD than CK is, is equal to CK: which is impossible by Part I. Therefore from C one straight line and one only CK can be drawn to AB equal to CE, lying on the opposite side of the perpendicular CD. Which was to be proved.

APPENDIX D.

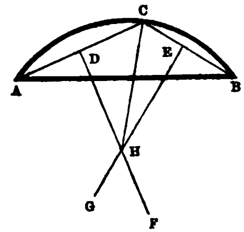
The problem of describing a circle of which a segment or an arc is given, may be solved by the following method, which is simpler than that given in the text.

BK. III. PROP. XXV. PROB.

A segment or an arc of a circle being given, to describe the circle of which it is the segment or the arc.

Let ACB be the given segment, or ACB the given arc of a circle. It is required to describe the circle of which it is the segment, or the arc.

In the arc ACB take any point C, and join CA, CB. Bisect (i. 10) CA, CB in D, E; and from D, E draw (i. 11) DF, EG perpendicular to AC, CB. Then as in the constⁿ of Bk. IV. Prop. V. it may be shewn that DF, EG being continually produced will meet. Let them be produced to meet in H; and join HC.



Then because DF is drawn bisecting at right angles the straight line in the circle AC, therefore the centre of the circle is in this line DF produced (iii. 1. Cor.); for like reason the centre of the circle is in EG. Hence the centre of the circle is a point in DF, and a point in EG, and must therefore be H, the point of intersection of DF, EG: therefore if with centre H and radius HC a circle be described, it will be the circle of which ACB is the segment, or ACB the arc. Which was to be done.

APPENDIX E.

The following propⁿˢ requiring the use of the 11th Book, are analogous to the one proved in Appendix C., and they may be called Propⁿˢ A. and B. of Bk. XI.

BK. XI. PROP. A. THEOR.

Of all the straight lines drawn to meet a plane from a point without it, that which is perpendicular to the plane shall be the shortest.

Let A be any point without the plane KL (see figure of Bk. XI. Prop. XII.); AF the perpendicular from A on the plane. Then of all the straight lines drawn from A to meet the plane, AF shall be the least.

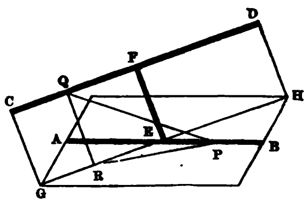
Take AD any straight line, meeting the plane in D; and join DF.

Then since AF is by hyp^s perpendicular to the plane, and DF meets it in F in that plane, therefore by the def^n of a straight line being perpendicular to a plane (XI. Def. 3), AFD is a right angle, and therefore the squares of AF, FD are equal (I. 47) to the square of AD. Hence the square of AF alone is less than the square of AD, and therefore AF less than AD. Similarly it may be shewn that AF is less than every other straight line drawn from A to meet the plane; therefore AF is the least of all such lines. Which was to be proved.

BK. XI. PROP. B. THEOR.

If there be two straight lines not in the same plane: then of all the straight lines joining any point in the one with any point in the other, the least shall be that which is perpendicular to each of them.

Let AB, CD be two straight lines not in the same plane; EF the straight line which is perpendicular to each of them. Then of all the straight lines joining any point in AB with any point in CD, EF shall be the least.



Take any straight line PQ joining P in AB with Q in CD. In the plane through CD and FE, draw through E (I. 31) GEH parallel to CFD, and from Q draw QR perpendicular to GH; join RP, which will be in the plane GH, through AB and GH.

Then since FE cuts the parallels CD, GH in F, E, the two interior angles on the same side CFE, FEG are equal (I. 29) to two right angles; and CFE is a right angle by const^n: therefore the other FEG is a right angle. And FE by hyp^s is at right angles to AB; therefore FE is at right angles to each of AB, GH at the point of their intersection E. Therefore EF is perpendicular to the plane

through AB, GH; and QR by const^n is parallel to EF: therefore QR is perpendicular to the plane GH. But PR is drawn meeting it in this plane; therefore by the def^n (xi. Def. 3) of a straight line being perpendicular to a plane, QRP is a right angle. Hence the square of PQ is equal to the squares of QR, RP (i. 47), that is, of FE, RP, since QREF is a parallelogram by const^n, and QR therefore (i. 34) equal to EF: therefore the square of FE alone is less than the square of PQ, and therefore FE than PQ. Similarly it may be shewn that FE is less than every other straight line drawn joining any point in AB with any point in CD. Hence EF is the least of all such lines. Which was to be proved.

THE END.

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